STOCHASTIC DYNAMICAL SYSTEMS EXERCISES

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by

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Exercise 1. We consider a sequence of i.i.d. \mathbb{R} -valued random variables $(Y_n)_{n\geq 1}$ with distribution μ . We are interested in the fluctuations of the random walk $(S_n)_{n\geq 0}$ defined by

$$S_0 = 0$$
 and $S_n = Y_1 + \ldots + Y_n$.

We introduce the ascending ladder epochs $T_n^+, n \ge 0$, defined by $T_0^+ = 0$ and, for $n \ge 1$,

$$T_n^+ := \inf\{k > T_{n-1}^+ : S_k > S_{T_{n-1}^+}\}$$

with the convention $\inf \emptyset = +\infty$ (in particular, it yields $T_n^+ = +\infty \Longrightarrow T_k^+ = +\infty$ for any $k \ge n$). These random variables take values in the set $\{1, 2, \ldots\} \bigcup \{+\infty\}$ and are stopping times with respect to the filtration $(\sigma(Y_1,\ldots,Y_n))_{n\geq 1}$ associated with the sequence $(Y_n)_{n\geq 1}$. When the random variables $T_n^+, n\geq 1$, are finite \mathbb{P} -a.s. (which occurs under some suitable conditions detailed in the course), we may consider the random variable $S_{T_1^+}$ defined by

$$S_{T_1^+} := \sum_{n=1}^{+\infty} S_n \ \mathbf{1}_{[T_1^+ = n]}$$

For any $n \ge 1$, we set $\tau_n^+ := T_n^+ - T_{n-1}^+$ and $A_n := S_{T_n^+} - S_{T_{n-1}^+}$; consequently,

$$T_n^+ = \tau_1^+ + \ldots + \tau_n^+$$
 and $S_{T_n^+} = A_1 + \ldots + A_n$.

- 1. Prove that, for any $k \ge 1$, it holds $\mathbb{P}(\tau_1^+ = k) = \mathbb{P}[S_1 \le 0, \dots, S_{k-1} \le 0, S_k > 0].$
- 2. Prove that, for any $k, l \ge 1$, it holds $\mathbb{P}[\tau_1^+ = k, \tau_2^+ = l] = \mathbb{P}[\tau_1^+ = k] \times \mathbb{P}[\tau_1^+ = l].$
- 3. Conclude that τ_1^+ and τ_2^+ are i.i.d. random variables (with values in $\mathbb{N}^* \cup \{+\infty\}$). By the same argument, it can be proved that $(\tau_n^+)_{n\geq 1}$ is a sequence of i.i.d. random variables with values in $\mathbb{N}^* \cup \{+\infty\}$.
- 4. Assume that $\mathbb{P}[\tau_1^+ < +\infty] = 1$.
 - (a) Check that, for any $s \in \mathbb{R}$,

$$\mathbb{P}[A_1 \le s] = \begin{cases} 0 & \text{when } s \le 0\\ \sum_{n=1}^{+\infty} \mathbb{P}[T_1^+ = n, 0 < S_n \le s] & \text{when } t > 0. \end{cases}$$

(b) Prove that, for any $s, t \in \mathbb{R}$,

$$\mathbb{P}[A_1 \le s, A_2 \le t] = \mathbb{P}[A_1 \le s] \times \mathbb{P}[A_2 \le t].$$

(c) Conclude that A_1 and A_2 are i.i.d. random variables with values in \mathbb{R}^{*+} .

By the same argument, it can be proved that $(A_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with values in \mathbb{R}^* .

- 5. Assume now that $\mathbb{E}[|Y_1|] < +\infty$.
 - (a) Prove that $\mathbb{P}[\tau_1^+ < +\infty] = 1$ if and only if $\mathbb{E}[Y_1] \ge 0$.

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(b) Under the condition $\mathbb{E}[Y_1] \ge 0$, check that $\mathbb{P}[T_n^+ < +\infty] = 1$ for any $n \ge 1$ and that the so called "Wald's formula" holds:

$$\mathbb{E}[A_1] = \mathbb{E}[Y_1] \times \mathbb{E}[\tau_1^+].$$

Hint: we may consider the ratio $\frac{A_1 + \ldots + A_n}{n} = \frac{S_{T_n^+}}{T_n^+} \times \frac{T_n^+}{n}$ and apply the strong law of large numbers.

- (c) Prove that $\mathbb{E}[\tau_1^+] = +\infty$ when the Y_i are centered.
- (d) Now, we assume $\mathbb{E}[Y_1] > 0$ and introduce the random variable $\tau_1^- := \inf\{k \ge 1 : S_k \le 0\}$. Using the strong law of large number, check that $\mathbb{P}[\tau_1^- = +\infty] > 0$ and deduce that $\mathbb{E}[\tau_1^+] < +\infty$. Conclude that $\mathbb{E}[A_1] < +\infty$.

Exercise 2. We consider a queue with one server and denote $\mathcal{A}_1, \mathcal{A}_2, \ldots$ the inter-arrival times between two successive customers; the arrival times of the customers are $0, \mathcal{A}_1, \mathcal{A}_1 + \mathcal{A}_2, \ldots$ We denote $\mathcal{S}_1, \mathcal{S}_2, \ldots$ the service time of the different customers.

We assume that $(\mathcal{A}_n)_{n\geq 1}$ and $(\mathcal{S}_n)_{n\geq 1}$ are two independent sequences of i.i.d random variables and that the distribution of the \mathcal{A}_n (resp. the \mathcal{B}_n) is exponential with parameter $\mathfrak{a} > 0$ (resp. with parameter $\mathfrak{s} > 0$).

We set $W_0 = 0$, and, for any $n \ge 1$, we denote W_n the waiting time of the n^{th} customer in the queue.

If the n^{th} customer arrives at time t, he is served at time $t + W_n$ and leaves the queue at time $t + W_n + S_n$; the customer n + 1 arrives at time $t + A_{n+1}$ and his waiting time W_{n+1} in the queue equals

$$W_{n+1} := \begin{cases} 0 & \text{when} \\ W_n + \mathcal{S}_n - \mathcal{A}_{n+1} & \text{otherwise.} \end{cases} \mathcal{A}_{n+1} \ge W_n + \mathcal{S}_n$$

In other words, setting $Y_{n+1} := \mathcal{A}_{n+1} - \mathcal{S}_n$, it holds $W_{n+1} := \max(W_n - Y_{n+1}, 0)$.

- 1. Check that $(W_n)_{n\geq 0}$ is a stochastic dynamical system (SDS) as defined in the course (give explicitly the family of random maps which defines this SDS).
- 2. Check that the distribution of the Y_n has the following density $t \mapsto h(t)$ with respect to the Lebesgue measure on \mathbb{R} :

$$\forall t \in \mathbb{R} \qquad h(t) := \frac{\mathfrak{a}}{\mathfrak{a} + \mathfrak{s}} \mathfrak{s} e^{\mathfrak{s} t} \mathbf{1}_{\mathbb{R}^{-}}(t) + \frac{\mathfrak{s}}{\mathfrak{a} + \mathfrak{s}} \mathfrak{a} e^{-\mathfrak{a} t} \mathbf{1}_{\mathbb{R}^{+}}(t).$$

- 3. Compute $\mathbb{E}[\mathcal{A}_n]$ and $\mathbb{E}[\mathcal{S}_n]$. Deduce that
 - (a) If $\mathfrak{a} > \mathfrak{s}$, then the process $(W_n)_{n>0}$ is positive recurrent.
 - (b) If $\mathfrak{a} = \mathfrak{s}$, then the process $(W_n)_{n \ge 0}$ is null recurrent.
 - (c) If $\mathfrak{a} < \mathfrak{s}$, then the sequence $(W_n)_{n \geq 0}$ tends \mathbb{P} -a.s. towards $+\infty$.
- 4. We denote τ the first time at which the queue is empty. In which cases is the random variable τ finite P-a.s.? has finite expectation?

Exercise 3. We consider the reflected random walk $(R_n)_{n\geq 0}$ on \mathbb{R}^+ defined by: $R_0 = x$ for some fixed $x \in \mathbb{R}^+$ and, for any $n \geq 0$,

$$R_{n+1} = |R_n - Y_{n+1}|$$

where $(Y_k)_{k\geq 1}$ is a sequence of i.i.d. \mathbb{R} -valued random variables whose distribution is adapted on \mathbb{R} . We assume that $\mathbb{E}[|Y_1|] < +\infty$ and $\mathbb{E}[Y_1] > 0$.

- 1. Does there exists an invariant probability measure on \mathbb{R}^+ for $(R_n)_{n\geq 0}$? Is it unique? Why?
- 2. Is $(R_n)_{n\geq 0}$ transient? positive recurrent? null recurrent?
- 3. We assume that the Y_i are uniformly distributed on the interval [0, C] for some C > 0. Compute explicitly the invariant probability measure. What is its support?
- 4. We assume that the Y_i have the exponential distribution with parameter $\lambda > 0$. Compute explicitly the invariant probability measure of $(R_n)_{n\geq 0}$ in this case. What is its support?

Exercise 4. We consider the evolution over the time of a life insurance contract. For any $n \ge 1$, a payment b_n is made in the life insurance portfolio at the beginning of the *n*-th period; we set $X_0 = 0$ and denote X_n the amount which has been accumulated during the *n* first periods. The amount X_{n-1} is subject to interest, given by the random variables $a_n = 1 + \delta_n$ (the coefficient δ_n is the interest rate announced by the insurance company). Thus, the value of the portfolio at the beginning of the *n*-th period is given by the equation

$$\begin{cases} X_0 &= 0\\ X_n &= a_n X_{n-1} + b_n = (1 + \delta_n) X_{n-1} + b_n & \text{when } n \ge 1. \end{cases}$$

Thus, it holds $X_n = F_n \circ \ldots \circ F_1(0) = L_n(0)$ where $F_n(x) := a_n x + b_n = (1 + \delta_n)x + b_n$ and $L_n = F_n \circ \ldots \circ F_1$.

The coefficients a_n and b_n may vary randomly over time, with some homogeneous behavior; we assume that the random variables (a_n, b_n) are i.i.d. with distribution μ on $\mathbb{R}^+ \times \mathbb{R}$.

• during each period, the interest rate may be equal to $\alpha > 0$ with probability $0 and to <math>-\beta \in]-1, 0[$ with probability q = 1 - p; in other words, the distribution of the random variable a_n equals $p\delta_{1+\alpha} + q\delta_{1-\beta}$, with p, q > 0, p+q = 1 and $0 < \alpha, \beta < 1$.

• the deposits b_n are non negative (i.e. the customer is not allowed to make withdrawals) and $\mathbb{E}[|\log b_n|] < +\infty$. We assume that

$$p\log(1+\alpha) + q\log(1-\beta) < 0. \tag{1}$$

- 1. Prove that, under the condition (1), the sequence $(F_1 \circ \ldots \circ F_n(x))_{n \ge 0}$ converges \mathbb{P} -a.s. towards a finite random variable Z_{∞} , which can be given explicitly and which does not depend on the value of x.
- 2. Deduce that the sequence $(X_n)_{n>0}$ converges in distribution to Z_{∞} .
- 3. The customer has signed the insurance contract because the announced mean interest rate $p(1 + \alpha) + q(1 \beta)$ satisfies the inequality

$$p(1+\alpha) + q(1-\beta) > 1.$$
 (2)

Check that, for any $0 < \beta < 1$ and $\alpha > 0$, both conditions (1) and (2) may hold simultaneously for some suitable choice of p and q to be specified.

Exercise 5. The nearest neighbourd (reflected or not) random walks on \mathbb{N}

Let $(Y_n)_{n\geq 0}$ be a sequence of independent \mathbb{Z} -valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with commun distribution $\mu = q\delta_{-1} + r\delta_0 + p\delta_1$ where $p, q, r \geq 0, p + q + r = 1$ and $p \times q \times r \neq 0$.

1. We first consider the classical random walk $(S_n)_{n>0}$ on \mathbb{Z} defined by

$$S_0 := x \in \mathbb{Z}$$
 and $S_n := x + Y_1 + \ldots + Y_n$ for $n \ge 1$.

- (a) Check that $(S_n)_{n\geq 0}$ is a stochastic dynamical system as defined in the course (give explicitly the family of random maps which defines this SDS).
- (b) What is the transition matrix of this Markov chain?
- (c) Show that $(S_n)_{n>0}$ is irreducible on \mathbb{Z} . Is this Markov chain aperiodic?
- (d) Let $\lambda = (\lambda_k)_{k \in \mathbb{Z}}$ be a measure on \mathbb{Z} (i.e. $\lambda_k \in \mathbb{R}$ for any $k \in \mathbb{Z}$) which is invariant for $(S_n)_{n \ge 0}$; prove that, for any $k \in \mathbb{Z}$,

$$q\lambda_{k+1} + r\lambda_k + p\lambda_{k-1} = \lambda_k. \tag{3}$$

Rewriting (3) as $q\lambda_{\ell+2} - (p+q)\lambda_{\ell+1} + p\lambda_{\ell} = 0$ for any $\ell \in \mathbb{Z}$, deduce that there exist constants $a, b \in \mathbb{R}$ such that $\lambda_{\ell} = a\left(\frac{p}{q}\right)^{\ell} + b$ for any $\ell \in \mathbb{Z}$.

Check that, in particular, the measure m on \mathbb{Z} such that m(k) = 1 for any $k \in \mathbb{Z}$ is always invariant for the random walk $(S_n)_{n\geq 0}$, whatever the values of p, q, r are.

(e) Assume p > q > 0 (the case when q > p > 0 can be studied similarly).

• Prove that $\lim_{n \to +\infty} S_n = +\infty$ and that $(S_n)_{n \ge 0}$ is transient on \mathbb{Z} .

- (f) Assume now p = q > 0. Set $\tau_1^+ = \inf\{n \ge 1 \mid S_n = 1\}$ and $\tau_1^- = \inf\{n \ge 1 \mid S_n = -1\}$.
 - Using the fact that the law μ is symmetric, check that $\mathbb{P}[\tau_1^+ < +\infty] = \mathbb{P}[\tau_1^- < +\infty] = 1$.
 - Prove that $\mathbb{P}[T_n^+ < +\infty] = \mathbb{P}[T_n^- < +\infty] = 1$ for any $n \ge 0$ (we use the same notations as in the course).
 - Check that $\mathbb{P}[S_{\tau_1^+} = 1] = \mathbb{P}[S_{\tau_1^-} = -1] = 1$ and that $\mathbb{P}[S_{T_n^+} = n] = \mathbb{P}[S_{T_n^-} = -n] = 1$ for any $n \ge 0$.
 - Deduce that $\liminf_{n \to +\infty} S_n = -\infty$ and $\limsup_{n \to +\infty} S_n = +\infty$ P-almost surely and that $(S_n)_{n \ge 0}$ is recurrent on \mathbb{Z} .
 - What is the unique (up to a multiplicative constant) σ -finite invariant measure for $(S_n)_{n>0}$ in this case?
 - Does there exist an unique (up to a multiplicative constant) invariant measure for $(S_n)_{n>0}$ in this case?
- 2. Now, we consider the reflected random walk $(A_n)_{n>0}$ on \mathbb{N} with absorption at 0 defined by

 $A_0 := x \in \mathbb{N}$ and $A_{n+1} := \max(0, A_n - Y_{n+1})$ for $n \ge 1$.

- (a) Check that $(A_n)_{n\geq 0}$ is a stochastic dynamical system as defined in the course (give explicitly the family of random maps which defines this SDS).
- (b) What is the transition matrix of this Markov chain?
- (c) Show that $(A_n)_{n>0}$ is irreducible on N. Is this Markov chain aperiodic?
- (d) Let $\lambda = (\lambda_k)_{k \in \mathbb{N}}$ be an invariant measure on \mathbb{N} for $(A_n)_{n \geq 0}$; prove that

$$(p+r)\lambda_0 + p\lambda_1 = \lambda_0$$
 and $p\lambda_{k+1} + r\lambda_k + q\lambda_{k-1} = \lambda_k$ for $k \ge 1$. (4)

Rewriting (4) as $p\lambda_1 - q\lambda_0 = 0$ and $p\lambda_{\ell+2} - (p+q)\lambda_{\ell+1} + q\lambda_\ell = 0$ for any $\ell \ge 0$, deduce that there exists a constant $a \in \mathbb{R}$ such that $\lambda_{\ell} = a \left(\frac{q}{p}\right)^{\ell}$ for any $\ell \in \mathbb{N}$.

- (e) Assume p > q > 0. Using the fact that the measure λ is finite in this case, deduce that $(A_n)_{n \ge 0}$ is positive recurrent.
- (f) Assume p = q > 0. Using the fact that $\limsup S_n = +\infty$ P-a.s., prove that $\mathbb{P}_x(\exists n \ge 1 \mid A_n = 0) = 1$ for any $x \in \mathbb{N}^{(2)}$. Deduce that $\mathbb{P}_x[A_n = 0 \ i.o.] = 1$.

Is $(A_n)_{n\geq 0}$ recurrent or transient in this case? And if it is recurrent, is it positive or null recurrent?

(g) Assume q > p > 0. Prove that $A_n \ge -S_n \mathbb{1}_{[S_n \le 0]}$ for any $n \ge 0$; deduce that $\lim_{n \to +\infty} A_n = +\infty$ \mathbb{P}_x -a.s. for any $x \in \mathbb{N}$.

Is $(A_n)_{n>0}$ recurrent or transient in this case?

3. Lastly, we consider the reflected random walk $(R_n)_{n>0}$ on \mathbb{N} with elastic collision at 0 defined by

 $R_0 := x \in \mathbb{N}$ and $R_{n+1} := |R_n - Y_{n+1}|$ for $n \ge 1$.

- (a) Check that $(R_n)_{n>0}$ is a stochastic dynamical system as defined in the course (give explicitly the family of random maps which defines this SDS).
- (b) What is the transition matrix of this Markov chain?
- (c) Show that $(R_n)_{n\geq 0}$ is irreducible on N. Is this Markov chain aperiodic?
- (d) Let $\lambda = (\lambda_k)_{k \in \mathbb{N}}$ be an invariant measure on \mathbb{N} for $(R_n)_{n>0}$; prove that

 $r\lambda_0 + p\lambda_1 = \lambda_0, \qquad p\lambda_2 + r\lambda_1 + (p+q)\lambda_0 = \lambda_1 \qquad \text{and} \qquad p\lambda_{k+1} + r\lambda_k + q\lambda_{k-1} = \lambda_k \quad \text{for} \quad k \ge 2.$ (5)Rewriting (5) as $p\lambda_1 - (p+q)\lambda_0 = 0$, $p\lambda_2 - (p+q)\lambda_1 + (p+q)\lambda_0 = 0$ and $p\lambda_{\ell+2} - (p+q)\lambda_{\ell+1} + q\lambda_{\ell} = 0$ for any $\ell \geq 1$, deduce that there exists a constant $a \in \mathbb{R}$ such that $\lambda_{\ell} = a \left(\frac{q}{p}\right)^{\ell}$ for any $\ell \in \mathbb{N}$.

- (e) Assume p > q > 0. Using the fact that the measure λ is finite in this case, deduce that $(R_n)_{n>0}$ is positive recurrent.
- (f) Assume p = q > 0. Using the fact that $\limsup S_n = +\infty$ and that the steps Y_n are ≤ 1 P-a.s., prove that $n \rightarrow +\infty$ $\mathbb{P}_x(\exists n \ge 1 \mid R_n = 0) = 1$ for any $x \in \mathbb{N}$. Deduce that $\mathbb{P}_x[R_n = 0 \text{ i.o.}] = 1$. Is $(R_n)_{n>0}$ recurrent or transient in this case? And if it is recurrent, is it positive or null recurrent?
- (g) <u>Assume q > p > 0</u>. Prove that $R_n \ge A_n$ for any $n \ge 0$; deduce that $\lim_{n \to +\infty} R_n = +\infty$ \mathbb{P}_x -a.s. for any $x \in \mathbb{N}$. Is $(R_n)_{n>0}$ recurrent or transient in this case?

² "i.o" means "infinitely often"; the event $[A_n = 0 \text{ } i.o.]$ also equals $\bigcap_{n \ge 1} \bigcup_{k \ge n} [A_k = 0] = \limsup_{n \to +\infty} [A_n = 0]$