Convergence and Counting in Infinite Measure

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Abstract. We construct non-uniform convergent lattices Γ of pinched, negatively curved Hadamard spaces, in any dimension $N \geq 2$. These lattices are *exotic*, by which we mean that they have a maximal parabolic subgroup $P < \Gamma$ such that $\delta(P) = \delta(\Gamma)$. We also give examples of divergent, non-uniform exotic lattices in dimension $N = 2$. Finally, we consider a particular class of such exotic lattices, with infinite Bowen-Margulis measure and whose cusps have a particular asymptotic profile (satisfying a "heavy tail condition"), and we give precise estimates of their orbital function; namely, we show that their orbital function is lower exponential with asymptotic behavior $\asymp \frac{e^{\delta_\Gamma R}}{R^{1-\kappa} L(R)}$, for a slowly varying function L.

1. Introduction

Let Γ be a *Kleinian group*, i.e. a discrete, torsionless group of isometries of a Hadamard space X of negative, pinched curvature $-B^2 \leq K_X \leq -A^2 < 0$, with quotient $\bar{X} = \Gamma\backslash X$. This paper is concerned with two mutually related problems :

1) The description of the distribution of the orbits of Γ on X, namely of fine asymptotic properties of the *orbital function* :

$$v_\Gamma(\mathbf{x}, \mathbf{y}; R) := \sharp\{\gamma \in \Gamma / d(\mathbf{x}, \gamma \cdot \mathbf{y}) \leq R\}$$

for $\mathbf{x}, \mathbf{y} \in X$. This has been the subject of many investigations since Margulis' [**27**] (see Roblin's book [**33**] and Babillot's report on [**1**] for a clear overview). The motivations to understand the behavior of the orbital function are numerous : for instance, a simple but important invariant is its *exponential growth rate*

$$\delta_\Gamma = \limsup_{R\to\infty} \frac{1}{R} \ln(v_\Gamma(\mathbf{x}, \mathbf{y}; R))$$

which has a major dynamical significance, since it coincides with the topological entropy of the geodesic flow when $\bar{X}$ is compact, and is related to many interesting rigidity results and characterization of locally symmetric spaces, cp. [**23**], [**9**], [**6**].

2) The pointwise behavior of *Poincaré series* associated with Γ :

$$P_\Gamma(\mathbf{x}, \mathbf{y}, s) := \sum_{\gamma\in\Gamma} e^{-sd(\mathbf{x},\gamma\cdot\mathbf{x})}, \qquad \mathbf{x}, \mathbf{y} \in X$$

at a neighborhood of $s = \delta_\Gamma$, which coincides with its *exponent of convergence*. The group Γ is said to be *convergent* if $P_\Gamma(\mathbf{x}, \mathbf{y}, \delta_\Gamma) < \infty$, and *divergent* otherwise. Divergence can also be understood in terms of dynamics as, by Hopf-Tsuju-Sullivan theorem, it is equivalent to ergodicity and total conservativity of the geodesic flow with respect to the Bowen-Margulis measure m_Γ on the unit tangent bundle $U\bar{X}$ (see again [**33**] for a complete account and a definition of m_Γ).

The regularity of the asymptotic behavior of v_Γ, in full generality, is well expressed in Roblin's results, which trace back to Margulis' work in the compact case :

Theorem 1.1 (Margulis [**27**] - Roblin [**32**], [**33**]).
Let X be a Hadamard manifold with pinched negative curvature and Γ a non elementary, discrete subgroup of isometries of X with non-arithmetic length spectrum [1], *then the exponential growth rate δ_Γ is a true limit and one gets :*

(1) if $\|m_\Gamma\| = \infty$ then $v_\Gamma(\mathbf{x}, \mathbf{y}; R) = o(e^{\delta_\Gamma R})$,

(2) if $\|m_\Gamma\| < \infty$, then $v_\Gamma(\mathbf{x}, \mathbf{y}; R) \sim \frac{||\mu_\mathbf{x}||.||\mu_\mathbf{y}||}{\delta_\Gamma \|m_\Gamma||} e^{\delta_\Gamma R}$,

where $(\mu_\mathbf{x})_{\mathbf{x}\in X}$ denotes the family of Patterson conformal densities of Γ, and m_Γ the Bowen-Margulis measure on $U\bar{X}$.

1. It means that the set $\mathcal{L}(\bar{X}) = \{\ell(\gamma) \ ; \ \gamma \in \Gamma\}$ of lengths of all closed geodesics of $\bar{X} = \Gamma\backslash X$ is not contained in a discrete subgroup of $\mathbb{R}$

Here, $f \sim g$ means that $f(t)/g(t) \to 1$ when $t \to \infty$; we will write $f \overset{c}{\asymp} g$ when $\frac{1}{c} \leq f(t)/g(t) \leq c$ for $c \geq 1$ and $t \gg 0$ (or just $f \asymp g$ when the constant c is not specified). The best asymptotic regularity to be expected is the existence of an equivalent, as in (ii) ; an explicit computation of the second term in the asymptotic development of v_Γ is a difficult question for locally symmetric spaces (and almost a hopeless question in the general Riemannian setting).

Theorem 1.1 shows that the key assumption for a regular behavior of v_Γ is that the Bowen-Margulis measure m_Γ is finite. This condition is clearly satisfied for *uniform lattices* Γ (i.e. when $\bar{X} = X/\Gamma$ is compact), and more generally for groups Γ such that m_Γ has compact support (e.g., *convex cocompact groups*), but it may fail for *nonuniform* lattices, that is when $\bar{X} = X/\Gamma$ has finite volume but is not compact.

The finiteness of m_Γ has a nice geometrical description in the case of *geometrically finite groups.* Recall that any orbit $\Gamma \cdot \mathbf{x}$ accumulates on a closed subset Λ_Γ of the geometric boundary ∂X of X, called the *limit set* of Γ ; the group Γ (or the quotient manifold $\bar{X}$) is said to be geometrically finite if Λ_Γ decomposes in the set of *radial limit points* (the limit points ξ which are approached by orbit points in the M-neighborhood of any given ray issued from ξ, for some $M < \infty$) and the set of *bounded parabolic points* (those ξ fixed by some parabolic subgroup P acting cocompactly on $\partial X \setminus \{\xi\}$) ; for a complete study of geometrical finiteness in variable negative curvature see [**12**] and Proposition 1.10 in [**33**], and for a description of their topology at infinity see [**15**]. A finite-volume manifold $\bar{X}$ is a particular case of geometrically finite manifold : it can be decomposed into a compact set and finitely many *cusps* $\bar{\mathcal{C}}_i$, i.e. topological ends of $\bar{X}$ of finite volume which are quotients of a horoball $\mathcal{H}_{\xi_i}$ centered at a bounded parabolic point $\xi_i \in \partial X$ by a maximal parabolic subgroup $P_i \subset \Gamma$ fixing ξ_i.

The principle ruling the regularity of the orbital function v_Γ of nonuniform lattices, as pointed out in [**13**] and in the following papers of the authors [**16**], [**17**], [**18**], is that the orbital functions v_{P_i} (defined in a similar way as v_Γ) capture the relevant information on the wildness of the metric inside the cusps, which in turn may imply $\|m_\Gamma\| = \infty$ and the irregularity of v_Γ. In this regard, distinctive properties of the group Γ and of its maximal parabolic subgroups P_i are their *type* (convergent or divergent, as defined above) and the *critical gap property* (CGP), i.e. if $\delta_{P_i} < \delta_\Gamma$ for all i. Actually, in [**13**] it is proved that, for geometrically finite groups Γ, the divergence of P_i implies $\delta_{P_i} < \delta_\Gamma$, and that the critical gap property implies that the group Γ is divergent, with $\|m_\Gamma\| < \infty$. On the other hand there exist geometrically finite groups Γ which do not satisfy the CGP : we call such groups *exotic*, and we say that a cusp is *dominant* if it is associated to a parabolic subgroup P with $\delta_P = \delta_\Gamma$. Geometrically finite, exotic groups may as well be convergent or divergent : in the first case, they always have $\|m_\Gamma\| = \infty$ (by Poincaré recurrence and Hopf-Tsuju-Sullivan theorem, as $\|m_\Gamma\| < \infty$ implies total conservativity) ; in the second case, in [**13**] it is proved that the finiteness of m_Γ depends on the convergence of the first moment series

$$\sum_{p \in P_i} d(x, px) e^{-\delta_\Gamma d(x,px)} < +\infty.$$

The main aim of this paper is to present examples of lattices Γ for which v_Γ has an irregular asymptotic behavior. According to our discussion, we will then focus on *exotic, non-uniform lattices.* The convergence property of exotic lattices is an interesting question on its own : while uniform lattices (as well as convex-cocompact groups) always are divergent, the only known examples of convergent groups, to the best of our knowledge, are given in [**13**] and have infinite covolume. The first result of the paper is to show that both convergent and divergent exotic lattices do exist. Actually, in Section 3, by a variation of the construction in [**13**] we obtain :

Theorem 1.2. *For any $N \geq 2$, there exist N-dimensional, finite volume manifolds of pinched negative curvature whose fundamental group Γ is (exotic and) convergent.*

Constructing exotic, divergent lattices is more subtle. We prove in Section 5 :

THEOREM 1.3. *There exist non compact finite area surfaces of pinched negative curvature whose group Γ is exotic and divergent.*

We stress the fact that the examples of Theorem 1.2 have infinite Bowen-Margulis measure ; on the other hand, the surfaces of Theorem 1.3 can have finite or infinite Bowen-Margulis measure, according to the prescribed behaviour of the metric in the cusps. Moreover, we believe that the assumption on the dimension in Theorem 1.3 is just technical, but at present we are not able to construct similar examples in dimension $N \geq 3$.

Finally, in Section 6 we address the initial question about how irregular the orbital function can be, giving estimates for the orbital function of a large family of exotic lattices with infinite Bowen-Margulis measure :

THEOREM 1.4. *Let $\kappa \in]1/2, 1[$. There exist non compact finite area surfaces with pinched negative curvature whose fundamental group Γ satisfies the following asymptotic property : for any $\mathbf{x}, \mathbf{y} \in X$*

$$v_\Gamma(\mathbf{x}, \mathbf{y}; R) \quad \asymp \quad \frac{e^{\delta_\Gamma R}}{R^{1-\kappa} L(R)} \qquad \textit{as} \quad R \to +\infty$$

for some slowly varying function[2] $L : \mathbb{R}^+ \to \mathbb{R}^+$.

In Section 6 we give a general, but more technical, result on the orbital function of divergent exotic lattices whose cusps have a asymptotic profile satisfying a "heavy tail condition" (see Theorem 6.1 and conditions $\mathbf{H}_0 - \mathbf{H}_3$). As far as we know, except for some precise asymptotic formulas established by Pollicott and Sharp [**31**] for the orbital function of normal subgroups Γ of a cocompact Kleinian group (hence, groups which are far from being geometrically finite or with finite Bowen-Margulis measure), these are the only examples of such precise asymptotic behavior for the orbital function of Kleinian groups with infinite Bowen-Margulis measure.

Remark. This work should be considered as a companion paper to [**16**] & [**18**], where we study the asymptotic properties of the integral version of v_Γ, i.e. the *growth function* of X :

$$v_X(\mathbf{x}; R) := vol_X(B(\mathbf{x}, R)).$$

In [**18**], we obtain optimal conditions on the geometry on the cusps in order that there exists a *Margulis function*, that is a Γ-invariant function $c : X \to \mathbb{R}^+$ such that

$$v_X(\mathbf{x}; R) \sim c(\mathbf{x}) e^{\omega_X R} \qquad \text{for } R \to +\infty$$

where ω_X is the exponential growth rate of the function v_X (the integral analogue of δ_Γ). Notice that ω_X can be different from δ_Γ, also for (non uniform) lattices, as we showed in [**16**].

2. Geometry of negatively curved manifolds with finite volume

2.1. Landscape. Additionally to those given in the introduction, we present here notations and familiar results about negatively curved manifolds. Amongst good references we suggest [**7**], [**20**], [**4**] and, more specifically related to this work, [**24**] and [**16**]. In the sequel, $\bar{X} = X/\Gamma$ is a N-dimensional complete connected Riemannian manifold with metric g whose sectional curvatures satisfy : $-B^2 \leq K_X \leq -A^2 < 0$ for fixed constants A and B ; we will assume $0 < A \leq 1 \leq B$ since in most examples g will be obtained by perturbation of a hyperbolic one and the curvature will equal -1 on large subsets of $\bar{X}$.

The family of normalized distance functions :

$$d(\mathbf{x}_0, \mathbf{x}) - d(\mathbf{x}, \cdot)$$

converges uniformly on compacts to the *Busemann function* $\mathcal{B}_\xi(\mathbf{x}_0, \cdot)$ for $\mathbf{x} \to \xi \in \partial X$. The *horoballs* $\mathcal{H}_\xi$ (resp. the *horospheres* $\partial\mathcal{H}_\xi$) centered at ξ are the sup-level sets (resp. the

2. A function $L(t)$ is said to be "slowly varying" or "of slow growth" if it is positive, measurable and $L(\lambda t)/L(t) \to 1$ as $t \to +\infty$ for every $\lambda > 0$.

level sets) of the function $\mathcal{B}_\xi(\mathbf{x}_0,\cdot)$. Given a horosphere $\partial\mathcal{H}_\xi$ passing through a point $\mathbf{x}$, we also set, for all $t \in \mathbb{R}$,

$$\mathcal{H}_\xi(t) := \{\mathbf{y}/\mathcal{B}_\xi(\mathbf{x}_0,\mathbf{y}) \geq \mathcal{B}_\xi(\mathbf{x}_0,\mathbf{x}) + t\}$$

resp. $\partial\mathcal{H}_\xi(t) := \{\mathbf{y}/\mathcal{B}_\xi(\mathbf{x}_0,\mathbf{y}) = \mathcal{B}_\xi(\mathbf{x}_0,\mathbf{x}) + t\}$. We will refer to $t = \mathcal{B}_\xi(\mathbf{x}_0,\mathbf{y}) - \mathcal{B}_\xi(\mathbf{x}_0,\mathbf{x})$ as to the *height* of $\mathbf{y}$ (or of the horosphere $\partial\mathcal{H}_\xi(t)$) in $\mathcal{H}_\xi$. Also, when no confusion is possible, we will drop the index $\xi \in \partial X$ denoting the center of the horoball. Recall that the Busemann function satisfies the fundamental cocycle relation

$$\mathcal{B}_\xi(\mathbf{x},\mathbf{z}) = B_\xi(\mathbf{x},\mathbf{y}) + B_\xi(\mathbf{y},\mathbf{z})$$

which will be crucial in the following.

An origin $\mathbf{x}_0 \in X$ being fixed, the Gromov product between $x, y \in \partial X \cong \mathbb{S}^{n-1}$, $x \neq y$, is defined as

$$(x|y)_{\mathbf{x}_0} = \frac{\mathcal{B}_x(\mathbf{x}_0,\mathbf{z}) + \mathcal{B}_y(\mathbf{x}_0,\mathbf{z})}{2}$$

where $\mathbf{z}$ is any point on the geodesic (x, y) joining x to y ; then, for any $0 < \kappa^2 \leq A^2$ the expression

$$D(x,y) = e^{-\kappa(x|y)_{\mathbf{x}_0}}$$

defines a distance on ∂X, cp [**10**]. Recall that for any $\gamma \in \Gamma$ one gets

$$D(\gamma\cdot x, \gamma\cdot y) = e^{-\frac{\kappa}{2}\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)}e^{-\frac{\kappa}{2}\mathcal{B}_y(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)}D(x,y). \tag{1}$$

In other words, the isometry γ acts on $(\partial X, D)$ as a conformal transformation with coefficient of conformality $|\gamma'(x)| = e^{-\kappa\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)}$ at x, since equality (1) may be rewritten

$$D(\gamma\cdot x), \gamma\cdot y) = \sqrt{|\gamma'(x)||\gamma'(y)|}\, D(x,y). \tag{2}$$

Recall that Γ is a torsion free nonuniform lattice acting on X by hyperbolic or parabolic isometries. For any $\xi \in \partial X$, denote by $(\psi_{\xi,t})_{t\geq 0}$ the radial semi-flow defined as follows : for any $\mathbf{x} \in X$, the point $\psi_{\xi,t}(\mathbf{x})$ lies on the geodesic ray $[\mathbf{x},\xi)$ at distance t from $\mathbf{x}$. By classical comparison theorems on Jacobi fields (see for instance [**24**]), the differential of $\psi_{\xi,t} : \partial\mathcal{H}_\xi \to \partial\mathcal{H}_\xi(t)$ satisfies $e^{-Bt}||v|| \leq ||d\psi_{\xi,t}(v)|| \leq e^{-At}||v||$ for any $t \geq 0$ and any vector v in the tangent space $T(\partial\mathcal{H}_\xi)$; consequently, if μ_t is the Riemannian measure induced on $\partial\mathcal{H}_\xi(t)$ by the metric of X, we have, for any Borel set $F \subset \partial\mathcal{H}_\xi$

$$e^{-B(N-1)t}\mu_0(F) \leq \mu_t(\psi_{\xi,t}(F)) = \int_F |Jac(\psi_{\xi,t})(x)|d\mu_0(x) \leq e^{-A(N-1)t}\mu_0(F).$$

As $\bar{X} = \Gamma\backslash X$ is non compact and $vol(\bar{X}) < \infty$, the manifold $\bar{X}$ can be decomposed into a disjoint union of a relatively compact subset $\bar{\mathcal{K}}$ and finitely many cusps $\bar{\mathcal{C}}_1, ..., \bar{\mathcal{C}}_l$, each of which is a quotient of a horoball $\mathcal{H}_{\xi_i}$, centered at some boundary point ξ_i, by a maximal parabolic subgroup P_i. As a consequence of Margulis' lemma, we can choose the family $(\mathcal{H}_{\xi_i})_{1\leq i\leq l}$ so that any two Γ-translates of the $\mathcal{H}_{\xi_i}$ are either disjoint or coincide (cp. [**33**], Proposition 1.10) ; we call these $\mathcal{H}_{\xi_i}$ a *fundamental system of horospheres* for X. Accordingly, the Dirichlet domain $\mathcal{D}$ of Γ centered at the base point $\mathbf{x}_0$ can be decomposed into a disjoint union $\mathcal{D} = \mathcal{K} \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_l$, where $\mathcal{K}$ is a convex, relatively compact set containing $\mathbf{x}_0$ in its interior and projecting to $\bar{\mathcal{K}}$, and the $\mathcal{C}_i$ are connected fundamental domains for the action of P_i on $\mathcal{H}_{\xi_i}$, projecting to $\bar{\mathcal{C}}_i$. We let $\mathcal{S}_i = \mathcal{D} \cap \partial\mathcal{H}_{\xi_i}$ be the corresponding, relatively compact fundamental domain for the action of P_i on $\partial\mathcal{H}_{\xi_i}$, so that $\mathcal{C}_i = \mathcal{D} \cap \mathcal{H}_{\xi_i} \simeq \mathcal{S}_i \times \mathbb{R}_+$.

Fixing an end $\bar{\mathcal{C}}$, and omitting in what follows the index i, let μ_t be the Riemannian measure induced by the Riemannian metric on the horosphere $\partial\mathcal{H}_\xi(t)$ corresponding to $\bar{\mathcal{C}}$. In [**16**] we defined the *horospherical area function* associated with the cusp $\bar{\mathcal{C}}$ as :

$$\mathcal{A}(t) = \mu_t(P\backslash\partial\mathcal{H}_\xi(t)) = \mu_t(\psi_{\xi,t}(\mathcal{S})).$$

This function depends on the choice of the initial horosphere $\partial\mathcal{H}_\xi$ for the end $\bar{\mathcal{C}}$, and the following result shows that this dependance is unessential for our counting problem :

Proposition 2.1. [**16**] *There exists a constant* $c = c(A, B, diam(\mathcal{S}))$ *such that*

$$v_P(R) \overset{c}{\asymp} \frac{1}{\mathcal{A}(\frac{R}{2})}.$$

This weak equivalence is the key to relate the irregularity of the metric in the cusp to the irregular asymptotic behaviour of the orbital function of P. The second crucial step of our work will then be to describe precisely the contribution of v_P in the asymptotic behavior of v_Γ assuming $\delta_P = \delta_\Gamma$.

2.2. Cuspidal geometry. The strategy to construct examples with irregular orbital functions as in Theorems 1.2, 1.3 and 1.4 is to perturb in a suitable manner the metric of a finite volume hyperbolic manifold $\Gamma\backslash\mathbb{H}^N$ in a cuspidal end $P\backslash\mathcal{H}$. If $\mathcal{H} = \{\mathbf{y}/\mathcal{B}_\xi(\mathbf{x}_0, \mathbf{y}) \geq t_0\}$, the hyperbolic metric writes on $\mathcal{H} \simeq \partial\mathcal{H} \times \mathbb{R}_+ \equiv \mathbb{R}^N \times \mathbb{R}_+$ as $g = e^{-(t-t_0)}d\mathbf{x}^2 + dt^2$ in the horospherical coordinates $\mathbf{y} = (\mathbf{x}, t)$, where $d\mathbf{x}^2$ denotes the induced flat Riemannian metric of $\partial\mathcal{H}$ and $t = \mathcal{B}_\xi(\mathbf{x}_0, \mathbf{y}) - t_0$. We will consider a new metric g in $P\backslash\mathcal{H}_\xi$ whose lift to $\mathcal{H}$ writes, in the same coordinates, as

$$g = \tau^2(t)d\mathbf{x}^2 + dt^2.$$

We extend this metric by Γ-invariance to $\Gamma\mathcal{H}$ and produce a new Hadamard space (X, g) with quotient $\bar{X} = \Gamma\backslash X$. The new manifold $\bar{X}$ has again finite volume, provided that

$$\int_0^{+\infty} \tau^{N-1}(t)dt < \infty$$

and the end $\bar{\mathcal{C}} = P\backslash\mathcal{H}$ is a new cusp; we call the function τ the *analytic profile* of the cusp $\bar{\mathcal{C}}$. The horospherical area function $\mathcal{A}$ associated with the profile τ satisfies $\mathcal{A} \asymp \tau^{N-1}$; by Proposition 2.1, it implies that :

(a) the parabolic group P has critical exponent $\delta_P = \frac{(N-1)\omega_\tau}{2}$, for $\omega_\tau := \limsup_{R\to+\infty} \frac{1}{R}|\ln(\tau(R))|$,

(b) P is convergent if and only if $\displaystyle\int_0^{+\infty} \frac{e^{-\omega_\tau(N-1)t}}{\tau^{N-1}(t)}\mathrm{d}t < \infty$.

Also, notice that the sectional curvatures at $(\mathbf{x}, t)$ are given by $K_{(\mathbf{x},t)}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = -\left(\frac{\tau'}{\tau}\right)^2$ and $K_{(\mathbf{x},t)}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}) = -\frac{\tau''}{\tau}$ (see [**5**]).

In Sections 4, 5 and 6 we will apply this strategy to prescribe curvature and analytic profiles τ at certain depths, depending on additional real parameters a, b, η, defined as follows.
For any convex function $\tau : \mathbb{R} \to \mathbb{R}^+$ with $\int_0^{+\infty} \tau^{n-1}(t)dt < \infty$ and satisfying the conditions

$$\forall\, t \leq t_0 \quad \tau(t) = e^{-(t-t_0)} \tag{3}$$

$$A < \tau''/\tau < B \tag{4}$$

$$\omega_\tau = \limsup_{t\to+\infty} \frac{|\ln(\tau(t))|}{t} < B \tag{5}$$

we will set, $a \geq t_0$

$$\tau_a(t) = e^{-a}\tau(t-a) \qquad \text{for } t \in \mathbb{R}.$$

This profile defines a manifold $\bar{X}$ with a cusp $\bar{\mathcal{C}}$ which is hyperbolic at height less than a and then has (renormalized) profile equal to τ.
Moreover, given parameters $a \geq t_0$, $b \geq 0$ and $\eta \in]0, A[$, a straightforward calculus proves the existence of a profile $\tau_{a,b,\eta}$ such that

$$\begin{cases} A - \eta < \tau''_{a,b,\eta}(t)/\tau_{a,b,\eta}(t) < B + \eta & \text{for all } t, \\ \tau_{a,b,\eta}(t) = e^{-t} & \text{for } \ t \leq a, \\ \tau_{a,b,\eta}(t) = e^{-\omega_\tau t} & \text{for } \ t \in [\Delta + a, \Delta + a + b], \\ \tau_{a,b,\eta}(t) = e^{-(2\Delta+a+b)}\tau(t - (2\Delta + a + b)) & \text{for } \ t \geq 2\Delta + a + b. \end{cases}$$

for some constant $\Delta = \Delta(A, B, \eta) \gg 0$. The function $\tau_{a,b,\eta}$ defines a cusp $\bar{\mathcal{C}}$ which is hyperbolic till height a, with constant curvature $-\omega_\tau^2$ in a band of width b at height $\Delta + a$, and which then has asymptotic (suitable renormalized) profile τ after height $2\Delta + a + b$.

3. Proof of Theorem 1.2 : construction of convergent lattices

An example of manifold of negative curvature with infinite volume and whose fundamental group is convergent is due to Dal'Bo-Otal-Peigné [**13**]. We propose in this Section a variation of their argument to produce a convergent nonuniform *lattice*. We will consider a finite volume hyperbolic manifold $\Gamma\backslash\mathbb{H}^N$ with *one* cuspidal end $P\backslash\mathcal{H}$ and deform the metric in this end as explained before to obtain a new Hadamard space (X, g) such that the quotient $\bar{X} = \Gamma\backslash X$ has finite volume and a dominant cusp $\bar{\mathcal{C}} = P\backslash\mathcal{H}$ with a *convergent* parabolic group P, whose exponent δ_P :

(1) is greater than the Poincaré exponent of Γ acting on $\mathbb{H}^N$, that is $N-1$;

(2) equals the Poincaré exponent δ_Γ of Γ corresponding to the new metric.

For this, we choose τ satisfying the conditions (3), (4), (5) with $\delta_P = \frac{(N-1)\omega_\tau}{2} > (N-1)$ and $\int_0^{+\infty} \frac{e^{-\omega_\tau(N-1)t}}{\tau(t)^{N-1}}\mathrm{d}t < \infty$, and we consider the profile τ_a for some $a > 0$ to be precised. Remark that the first condition can be satisfied only if $\omega_\tau > 2$ which requires $B^2/A^2 > 4$. We will denote by d the distance on (X, g) corresponding to this new metric, and by d_0 the hyperbolic distance. We emphasize that the perturbation of the metric will not change neither the algebraic structure of the groups Γ and P, nor the horospheres $\mathcal{H}(t)$ (which are only modified in size and not as subsets of $\mathbb{H}^n$) and their radial flow ; however, the orbital function has different behavior before and after perturbation.

Now, we need to introduce a natural decomposition of geodesic segments according to their excursions in the cusp, which will enable us to encode elements of Γ by sequences of parabolic elements travelling far in the cusp and elements staying in a fixed, compact subset. We use the same notations as in 2.1 for the compact subset $\mathcal{K}$, the fundamental horosphere $\mathcal{H} = \mathcal{H}_{\xi_1}$ of X, and the Borel fundamental domain $\mathcal{S}$ be for the action of P on $\partial\mathcal{H}$. Let $h > c = \mathrm{diam}(\mathcal{K})$: for every $\gamma \in \Gamma$, the geodesic segment $[\mathbf{x}_0, \gamma\cdot\mathbf{x}_0]$ intersects $r = r(\gamma)$ disjoint translates $g.\mathcal{H}(h)$ (with the convention $r = 0$ if the interSection with $\bigcup_{g\in\Gamma} g.\mathcal{H}(h)$ is empty). In case $r \geq 1$, denote by $\mathbf{z}_1^-, \cdots, \mathbf{z}_r^-$ (resp. $\mathbf{z}_1^+, \cdots, \mathbf{z}_r^+$) the hitting (resp. exit) points of the oriented geodesic segment $[\mathbf{x}_0, \gamma\cdot\mathbf{x}_0]$ with translates of $\mathcal{H}(h)$ in this order. Hence we get

$$[\mathbf{x}_0, \gamma\cdot\mathbf{x}_0] \cap \left(\bigcup_{g\in\Gamma} g\cdot\mathcal{H}(h)\right) = [\mathbf{z}_1^-, \mathbf{z_1}^+] \cup \cdots \cup [\mathbf{z}_r^-, \mathbf{z}_r^+].$$

Accordingly, when $r \geq 1$ we can define the points $\mathbf{y}_1^-, \mathbf{y}_1^+ \cdots \mathbf{y}_r^-, \mathbf{y}_r^+$ on $[\mathbf{x_0}, g\cdot\mathbf{x}_0]$ such that, for any $1 \leq i \leq r$, the geodesic segment $[\mathbf{y}_i^-, \mathbf{y}_i^+]$ is the connected component of $[\mathbf{x_0}, \gamma\cdot\mathbf{x}_0] \cap \left(\bigcup_{g\in\Gamma} g\cdot\mathcal{H}\right)$ containing $[\mathbf{z}_i^-, \mathbf{z}_i^+]$. We also set $\mathbf{y}_0^+ = \mathbf{x}_0$ and $\mathbf{y}_{r+1}^- = \gamma\cdot\mathbf{x}_0$.
With these notations, there exist uniquely determined isometries $g_1, \cdots, g_r \in \Gamma$ and $p_1, \cdots, p_r \in P$ such that $\mathbf{y}_1^- \in g_1\cdot\mathcal{S}$, $\mathbf{y}_1^+ \in g_1p_1\cdot\mathcal{S}, \cdots, \mathbf{y}_r^+ \in g_1p_1\cdots g_rp_r\cdot\mathcal{S}$. Finally, we define g_{r+1} by the relation

$$\gamma = g_1p_1\cdots g_rp_rg_{r+1}$$

which we call the *horospherical decomposition of* γ *at height* h. Notice that this decomposition depends only on the initial hyperbolic metric. We also set $\mathbf{x}_0^+ = \mathbf{x}_0, \mathbf{x}_{r+1}^- = \gamma\cdot\mathbf{x}_0$ and $\mathbf{x}_i^- = g_1p_1\cdots g_i\cdot\mathbf{x}_0, \mathbf{x}_i^+ = g_1p_1\cdots g_ip_i\cdot\mathbf{x}_0$ for $1 \leq i \leq r$. We then have :

Lemma 3.1. *Let* $\gamma = g_1p_1\cdots p_rg_{r+1}$ *be the horospherical decomposition of* γ *at height* h *:*
(i) for every $i \in \{1, \cdots, r+1\}$ *the geodesic segments* $[\mathbf{x}_{i-1}^+, \mathbf{x}_i^-]$ *and* $[\mathbf{x_0}, g_i\cdot\mathbf{x_0}]$ *lie outside the set* $\bigcup_{g\in\Gamma} g.\mathcal{H}(h+2c)$*, for* $c = \mathrm{diam}(\mathcal{K})$ *;*
(ii) for every $i \in \{1, \cdots, r\}$ *the geodesic segments* $[\mathbf{x}_0, p_i\cdot\mathbf{x}_0]$ *have length greater than* $2(h-c)$ *and intersect the set* $\bigcup_{g\in\Gamma} g.\mathcal{H}(h-c)$*.*

Proof. Assume $r \geq 1$ and fix $1 \leq i \leq r+1$. By construction, each geodesic segment $[\mathbf{y}_{i-1}^+, \mathbf{y}_i^-]$ lies outside $\bigcup_{g\in\Gamma} g.\mathcal{H}(h)$. Then, each segment $[\mathbf{x}_{i-1}^+, \mathbf{x}_i^-]$ lies outside $\bigcup_{g\in\Gamma} g.\mathcal{H}(h+2c)$ since $d(\mathbf{x}_{i-1}^+, \mathbf{y}_{i-1}^+)$ and $d(\mathbf{x}_i^-, \mathbf{y}_i^-)$ are both smaller than c. Since $[\mathbf{x}_{i-1}^+, \mathbf{x}_i^-] = g_1p_1\cdots g_{i-1}p_{i-1}\cdot[\mathbf{x_0}, g_i\cdot\mathbf{x_0}]$ and the set $\bigcup_{g\in\Gamma} g.\mathcal{H}(h+2c)$ is Γ-invariant, the same holds for the segment $[\mathbf{x_0}, g_i\cdot\mathbf{x_0}]$. To prove statement (2), notice that the segment $[\mathbf{y}_i^-, \mathbf{y}_i^+]$ intersects the set

$\bigcup_{g\in\Gamma} g.\mathcal{H}(h)$ and has endpoints in $\partial\mathcal{H}$, so that $d(\mathbf{y}_i^-, \mathbf{y}_i^+) \geq 2h$; one concludes using the facts that $d(\mathbf{x}_i^-, \mathbf{y}_i^-)$ and $d(\mathbf{x}_i^+, \mathbf{y}_i^+)$ are both smaller than c and that $[\mathbf{x}_i^-, \mathbf{x}_i^+] = g_1 p_1 \cdots p_{i-1} g_i \cdot [\mathbf{x_0}, p_i \cdot \mathbf{x_0}]$. □

Moreover, the distance function is almost additive, with respect to this decomposition :

Lemma 3.2. *There exists a constant $C \geq 0$ such that for all $\gamma \in \Gamma$ with horospherical decomposition $\gamma = g_1 p_1 \cdots g_r p_r g_{r+1}$ at height h :*

$$d(\mathbf{x}_0, \gamma \cdot \mathbf{x}_0) \geq \sum_{i=1}^{r+1} d(\mathbf{x}_0, g_i \cdot \mathbf{x}_0) + \sum_{i=1}^{r} d(\mathbf{x}_0, p_i \cdot \mathbf{x}_0) - rC.$$

Proof. With the above notations one gets, for $C := 4\mathrm{diam}(\mathcal{K})$

$$\begin{aligned} d(\mathbf{x}_0, \gamma \cdot \mathbf{x}_0) &= \sum_{i=1}^{r+1} d(\mathbf{y}_{i-1}^+, \mathbf{y}_i^-) + \sum_{i=1}^{r} d(\mathbf{y}_i^-, \mathbf{y}_i^+) \\ &\geq \sum_{i=1}^{r+1} d(\mathbf{x}_{i-1}^+, \mathbf{x}_i^-) + \sum_{i=1}^{r} d(\mathbf{x}_i^-, \mathbf{x}_i^+) - 4r\mathrm{diam}(\mathcal{K}) \\ &= \sum_{i=1}^{r+1} d(\mathbf{x}_0, g_i \cdot \mathbf{x}_0) + \sum_{i=1}^{r} d(\mathbf{x}_0, p_i \cdot \mathbf{x}_0) - rC. \qquad \square \end{aligned}$$

Now assume $h > c = \mathrm{diam}(\mathcal{K})$ and $a = h+2c$. Set $\Gamma_h = \{\gamma \in \Gamma / [\mathbf{x}_0, \gamma\cdot\mathbf{x}_0] \cap g.\mathcal{H}(h+2c) = \emptyset$ for all $g \in \Gamma\}$ and $P_h = \{p \in P / d(\mathbf{x}_0, p\cdot\mathbf{x}_0) \geq 2(h-c)\}$. Let $\gamma \in \Gamma$ with horospherical decomposition $\gamma = g_1 p_1 \cdots g_r p_r g_{r+1}$ at height h. By Lemma 3.1 (i), the geodesic segments $[\mathbf{x}_0, g_i \cdot \mathbf{x}_0], 1 \leq i \leq r+1$, stay outside $\bigcup_{g\in\Gamma} g \cdot \mathcal{H}(h+2c)$, thus outside the set where the metric is perturbed, so that $g_i \in \Gamma_h$ and $d(\mathbf{x}_0, g_i\cdot\mathbf{x}_0) = d_0(\mathbf{x}_0, g_i\cdot\mathbf{x}_0)$. Similarly, $p_i \in P_h$ by Lemma 3.1 (ii). Consequently, the Poincaré series of Γ for the perturbed metric is

$$\begin{aligned} P_\Gamma(\mathbf{x}_0, \delta) \leq & \sum_{\gamma\in\Gamma_h} e^{-\delta d(\mathbf{x}_0, \gamma\cdot\mathbf{x}_0)} + \sum_{p\in P_h} e^{-\delta d(\mathbf{x}_0, p\cdot\mathbf{x}_0)} + \sum_{r\geq 1} \sum_{\substack{g_1,\cdots,g_{r+1}\in\Gamma_h \\ p_1,\cdots,p_r\in P_h}} e^{-\delta d(\mathbf{x}_0, g_1 p_1 \cdots g_r p_r g_{r+1}\cdot\mathbf{x}_0)} \\ \leq & \sum_{\gamma\in\Gamma} e^{-\delta d_0(\mathbf{x}_0, \gamma\cdot\mathbf{x}_0)} + \sum_{p\in P_h} e^{-\delta d(\mathbf{x}_0, p\cdot\mathbf{x}_0)} + \sum_{r\geq 1} \left(e^{C\delta} \sum_{g\in\Gamma_h} e^{-\delta d(\mathbf{x}_0, g\cdot\mathbf{x}_0)} \sum_{p\in P_h} e^{-\delta d(\mathbf{x}_0, p\cdot\mathbf{x}_0)} \right)^r \\ \leq & \sum_{\gamma\in\Gamma} e^{-\delta d_0(\mathbf{x}_0, \gamma\cdot\mathbf{x}_0)} + \sum_{p\in P_h} e^{-\delta d(\mathbf{x}_0, p\cdot\mathbf{x}_0)} + \sum_{r\geq 1} \left(e^{C\delta} \sum_{\gamma\in\Gamma} e^{-\delta d_0(\mathbf{x}_0, \gamma\cdot\mathbf{x}_0)} \sum_{p\in P_h} e^{-\delta d(\mathbf{x}_0, p\cdot\mathbf{x}_0)} \right)^r \end{aligned}$$

For $\delta = \delta_P$ the term $\sum_{\gamma\in\Gamma} e^{-\delta d_0(\mathbf{x}_0, \gamma\cdot\mathbf{x}_0)} < +\infty$ since $\delta_P > N-1$. On the other hand, $\sum_{p\in P_h} e^{-\delta d(\mathbf{x}_0, p\cdot\mathbf{x}_0)}$ tends to 0 as $h \to +\infty$ (we emphasize that each term $e^{-\delta d(\mathbf{x}_0, p\cdot\mathbf{x}_0)}$ depends on $a = h+2c$). Indeed, by Proposition 2.1,

$$\begin{aligned} \sum_{p\in P_h} e^{-\delta_P d(\mathbf{x}_0, p\cdot\mathbf{x}_0)} &\preceq \sum_{n\geq 2(h-c)} e^{-\delta n} v_P(n) \asymp \sum_{n\geq 2(h-c)} \frac{e^{-\delta n}}{\mathcal{A}_a(\frac{n}{2})} \\ &\asymp \int_{2(h-c)}^{+\infty} \frac{e^{-\delta u}}{\tau_a(\frac{u}{2})^{N-1}} du \asymp e^{a(N-1)} \int_{2(h-c)}^{+\infty} \frac{e^{-\delta u}}{\tau(\frac{u}{2}-a)^{N-1}} du \\ &\asymp e^{h(N-1-2\delta)} \end{aligned}$$

with $\delta > N-1$. We may thus choose a large enough so that

$$e^{C\delta_P} \sum_{\gamma\in\Gamma} e^{-\delta_P d_0(\mathbf{x}_0, \gamma\cdot\mathbf{x}_0)} \sum_{p\in P_h} e^{-\delta_P d(\mathbf{x}_0, p\cdot\mathbf{x}_0)} < 1$$

which readily implies $P_\Gamma(\mathbf{x}_0, \delta_P) < +\infty$, hence $\delta_P \geq \delta_\Gamma$. As P is a subgroup of Γ, this implies that $\delta_P = \delta_\Gamma$, hence Γ is a convergent group. □

4. Critical gap property versus divergence

In this Section we start constructing a hyperbolic lattice Γ of $\mathbb{H}^2$ which is generated by suitable parabolic isometries, so that the resulting surface $\Gamma\backslash\mathbb{H}^2$ has finite volume. In the disk model of the hyperbolic plane, we choose $r \geq 2$ and $2r$ boundary points $\xi_0 = \eta_0, \xi_1, \eta_1, \cdots, \xi_r, \eta_r = \xi_0$ of $\mathbb{S}^1 = \partial\mathbb{B}^2$ in cyclic order, and consider (uniquely determined) parabolic isometries $p_1, \cdots, p_r$ such that for $1 \leq i \leq r$ we have $p_i \cdot \xi_i = \xi_i$ and $p_i \cdot \eta_{i-1} = \eta_i$. We remark that all η_i belong to the Γ orbit of the point η_0, which is a parabolic fixed point of the isometry $p_0 := p_r p_{r-1} \cdots p_1$.

Property 4.1. *The group $\Gamma = \langle p_1, \cdots, p_r\rangle$ is a free non abelian group over $p_1, \cdots, p_r$. The quotient $\Gamma\backslash\mathbb{B}^2$ is a finite surface with $r+1$ cuspidal ends, with a cusp $\bar{\mathcal{C}}_i$ for each parabolic subgroup $\mathcal{P}_i = \langle p_i\rangle$ for $i = 1, .., r$, and another cusp $\bar{\mathcal{C}}_0$ corresponding to the parabolic subgroup $\mathcal{P}_0 = \langle p_0\rangle$ fixing ξ_0.*

Each element $\gamma \in \Gamma \setminus \{\mathrm{Id}\}$ can be written in a unique way as a word with letters in the alphabet $\mathcal{A} := \{p_1^{\pm 1}, \cdots, p_r^{\pm 1}\}$; namely, one gets

$$\gamma = p_{j_1}^{\epsilon_1} \cdots p_{j_n}^{\epsilon_n} \tag{6}$$

with $p_{j_1}^{\epsilon_1}, \cdots, p_{j_n}^{\epsilon_n} \in \mathcal{A}$, $n \geq 1$ and with adjacent letters which are not inverse to each other. Such expression with respect to the natural (but not canonical) choice of the alphabet $\mathcal{A}$ is called a *coding* of elements of Γ. We will call j_1 is the *first index* of γ, denoted by i_γ ; similarly j_n the *last index* and is denoted by l_γ.

4.1. A new coding for elements of Γ. We code here the elements of Γ by blocks, with some admissibility rules to be precised. This new coding is designed to obtain a contraction property for an operator that will be introduced and studied in the next Sections whose restriction to some suitable space of functions present remarkable spectral properties. We first rewrite the decomposition (6) as follows

$$\gamma = p_{i_1}^{\ell_1} p_{i_2}^{\ell_2} \cdots p_{i_m}^{\ell_m} \tag{7}$$

with $m \geq 1, \ell_1, \cdots, \ell_m \in \mathbb{Z}^*$ and $i_j \neq i_{j+1}$ for $1 \leq j < m$. When all the $\ell_j, 1 \leq j \leq m$, belong to $\{\pm 1\}$, one says that γ is a *level 1 word*; the set of such words is denoted by $\mathcal{W}_1$. Then, we select all the $\ell_j, 1 \leq j \leq m$, with $|\ell_j| \geq 2$ and write γ as

$$\gamma = p_{j_0}^{l_0} Q_1 p_{j_1}^{l_1} Q_2 \cdots p_{j_{k-1}}^{l_{k-1}} Q_k p_{j_k}^{l_k} \tag{8}$$

for $k \geq 0$, with :

- $|l_1|, \cdots, |l_{k-1}| \geq 2$,
- $|l_0|, |l_k| \neq 1$,
- each Q_j is either empty or a level 1 word, with $i_{Q_j} \neq j_{i-1}$ and $l_{Q_j} \neq j_i$.

The decomposition by blocks (8) is still unique ; it only uses letters from the new alphabet

$$\mathfrak{B} := \widehat{\mathcal{P}}_1 \cup \cdots \cup \widehat{\mathcal{P}}_r \cup \mathcal{W}_1$$

where $\widehat{\mathcal{P}}_i := \{p_i^n / |n| \geq 2\}$ for $1 \leq i \leq r$, possibly with $p_{i_0}^{l_0} = 1$ or $p_{i_r}^{l_r} = 1$. Notice that this decomposition does not depend on the metric, but only on the presentation chosen for Γ. We will call *blocks* the letters of this new alphabet, and say that a word $\beta_1 \cdots \beta_m$ in the alphabet $\mathfrak{B}$ is *admissible* if the last letter of any block β_i is different from the first one of β_{i+1} for $1 \leq i \leq m-1$. So, any $\gamma \in \Gamma \setminus \{\mathrm{Id}\}$ can be written as a finite, admissible word $\beta_1 \cdots \beta_m$ on $\mathfrak{B}$; the ordered sequence of the β_i's is called the $\mathfrak{B}$-*decomposition* of γ and the number m of blocks is denoted by $|\gamma|_{\mathfrak{B}}$. Finally, we denote by $\Sigma_{\mathfrak{B}}$ the set of all finite admissible words with respect to $\mathfrak{B}$.

4.2. A new metric in the cusps. We consider a fundamental system of horoballs $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_r$ centered respectively at the parabolic points $\xi_0, \xi_1, \xi_2, \cdots, \xi_r$ and such that all the horoballs $\gamma \cdot \mathcal{H}_i$, for $\gamma \in \Gamma, 0 \leq i \leq r$, are disjoint or coincide, as in Section 2.1. Then, we modify the hyperbolic metric in the cuspidal ends $\bar{\mathcal{C}}_i = P_i\backslash\mathcal{H}_i$ as follows. We choose positive constants $a_0, a_1, \cdots, a_{r-1}, a_r, b$ and η, functions $\tau_0, \tau_1, \cdots, \tau_{r-1}$ and τ_r as in Section 2.2 such that

$$\omega_{\tau_r} = \max(\omega_{\tau_0}, \cdots, \omega_{\tau_r}) > 1$$

and we prescribe the profile τ_{i,a_i} for the i-th cusp $\bar{\mathcal{C}}_i$ for $0 \leq i \leq r-1$, and the profile $\tau_{r,a_r,b}$ on the last dominant cusp $\bar{\mathcal{C}}_r = \bar{\mathcal{C}}$. This yields a new surface $X = (\mathbb{B}^2, g_{a_0,\cdots,a_r,b})$, with quotient $\bar{X} = \Gamma\backslash X$ of finite area. Since the metric on X depends, in particular, on the value of the parameter b, which will play a crucial role in what follows, we shall denote the induced distances on X, $\partial X \cong \mathbb{S}^1$ and the conformal factor respectively by d_b, D_b and $|\cdot|_b$; on the other hand, we shall omit the index b in the Busemann function and in the Gromov product, to simplify notations. The dependence of D_b on the parameter b is described by the following lemma, whose proof can be found in [**30**] :

LEMMA 4.2. *Let $b_0 > 0$ be fixed. There exists $c \geq 1$ and $\alpha \in]0,1]$ such that the family of distances $(D_b)_{0\leq b\leq b_0}$, are Hölder equivalent ; namely for all $b \in [0, b_0]$ we have*

$$\frac{1}{c} D_0^{1/\alpha} \leq D_b \leq c D_0^{\alpha}.$$

4.3. Ping-pong by blocks. For any $1 \leq i \leq r$, we consider the sub-arcs $I_i := [\eta_{i-1}, \eta_i]$ and $I'_i := [p_i^{-1} \cdot \eta_{i-1}, p_i \cdot \eta_i]$ of $\mathbb{S}^1$ containing ξ_i. There exists a ping-pong dynamic between these intervals : namely, for any block $\beta \in \mathfrak{B}$, we have

- if $\beta \in \mathcal{W}_1$, then $\beta \cdot I'_i \subset I_{i_\beta}$ for any $i \neq l_\beta$
- if $\beta \in \widehat{\mathcal{P}}_l$ with $l = i_\beta = l_\beta$, then $\beta \cdot I_i \subset I'_l$ for any $i \neq l$.

Moreover, for any γ with $\mathfrak{B}$-decomposition $\beta_1 \cdots \beta_m$, we define a compact subset $K_\gamma \subset \mathbb{S}^1$ as follows :

- $K_\gamma = \cup_{i\neq l_\gamma} I'_i$, if $\beta_m \in \mathcal{W}_1$
- $K_\gamma = \cup_{i\neq l_\gamma} I_i$, if $\beta_m \in \widehat{\mathcal{P}}_l$ with $l = l_\gamma$.

Then, using the fact that the closure of the sets I'_i and $\partial X \setminus I_i$ are disjoint, one gets :

LEMMA 4.3. *There exists a constant $C = C(A, \eta) > 0$ such that*

$$d_b(\mathbf{x}_0, \gamma \cdot \mathbf{x}_0) - C \leq \mathcal{B}_x(\gamma^{-1} \cdot \mathbf{x}_0, \mathbf{x}_0) \leq d_b(\mathbf{x}_0, \gamma \cdot \mathbf{x}_0)$$

for any $\gamma \in \Gamma$ and any $x \in K_\gamma$

which yield to the

COROLLARY 4.4. *There exist a real number $r \in]0,1[$ and a constant $C > 0$ such that for any $\gamma \in \Gamma$ with length $|\gamma|_{\mathfrak{B}} = k$, one gets*

$$\forall x \in K_\gamma \qquad |\gamma'(x)|_0 \leq C r^k.$$

Proof. See Prop.2.2 in [**3**].

4.4. Coding for the limit points. An infinite word on the alphabet $\mathfrak{B}$, i.e. an infinite sequence $\bar{\beta} = (\beta_n)_{n\geq 1}$ of elements of $\mathfrak{B}$ is called *admissible* if any finite subword $\beta_1 \cdots \beta_k$ is admissible ; the set of such words is denoted by $\Sigma_{\mathfrak{B}}^+$. Corollary 4.4 implies in particular the following fundamental fact :

LEMMA 4.5. *For any $\bar{\beta} \in \Sigma_{\mathfrak{B}}^+$, the sequence $(\beta_1 \cdots \beta_n \cdot \mathbf{x}_0)_{n\geq 0}$ converges to some point $\pi(\bar{\beta}) \in \partial X$; the map $\pi : \Sigma_{\mathfrak{B}}^+ \to \partial X$ is one-to-one, and its image $\pi(\Sigma_{\mathfrak{B}}^+)$ coincides with the subset $\partial_0 X := \partial X \setminus \left(\bigcup_{i=1}^r \Gamma \cdot \xi_i \cup \Gamma \cdot W_1\right)$ where $W_1 := \overline{\mathcal{W}_1 \cdot \mathbf{x}_0} \cap \partial X$.*

Notice that, if γ has $\mathfrak{B}$-decomposition $b_1 \cdots b_m$, then the subset K_γ defined in 4.3 is the closure of the subset corresponding, via the coding map π, to the the infinite sequences $\bar{\beta} = (\beta_n)_{n\geq 1}$ such that the *concatenation* $\gamma * \bar{\beta} = (b_1 \cdots b_m \beta_1 \cdots \beta_i \cdots)$ is admissible. We also set $J_\gamma := Cl\{\pi(\gamma * \bar{\beta}) \mid \bar{\beta} \in K_\gamma\}$, that is the closure of points corresponding to admissible sequences obtained by concatenation with the $\mathfrak{B}$-decomposition of γ.

As indicated previously, this coding by blocks is of interest since the classical shift operator on $\Sigma_{\mathfrak{B}}^+$ induces locally, exponentially expanding maps T^n on $\partial_0 X$; the map T, described for instance in details in [**14**], has countably many inverse branches, each of them acting by contraction on some subset of ∂X. Namely, we consider on $\Sigma_{\mathfrak{B}}^+$ the natural shift θ defined by

$$\theta(\bar{\beta}) := (\beta_{k+1})_{k\geq 1}, \quad \forall \bar{\beta} = (\beta_k)_{k\geq 1} \in \Sigma_{\mathfrak{B}}^+$$

This map induces a transformation $T : \partial_0 X \to \partial_0 X$ via the coding π ; moreover, T can be extended to the whole ∂X by setting, for any γ with $\mathfrak{B}$-decomposition $\gamma = \gamma_1\gamma_2\cdots\gamma_n$ and $\xi \in \{\xi_1, \cdots, \xi_r\} \cup W_1$

$$T(\gamma.\xi) := \gamma_2 \cdots \gamma_n.\xi$$

and $T(\xi) := \xi$. Then, for every block $\beta \in \mathfrak{B}$, the restriction of T to J_β is the action by β^{-1} ; by the dynamic described above, the inverse branches of the map T have the following property

Property 4.6. *There exist* $0 < r < 1$ *and a constant* $C > 0$ *such that, for any* $\gamma \in \Gamma$ *with* $|\gamma|_{\mathfrak{B}} = k$ *and for* $x, y \in K_\gamma$ *we have*

$$D_0(\gamma \cdot x, \gamma \cdot y) \leq Cr^k D_0(x, y).$$

This property is crucial for the investigation of the spectral properties of the transfer operator, which will be introduced in the next Section.

5. Existence of divergent exotic lattices

This Section is devoted to prove Theorem 1.3 : there exist two dimensional exotic and divergent lattices. Existence of infinite covolume exotic and divergent discrete groups has already been provided in [**30**], by the following procedure.

Given a Schottky group $\Gamma =< h, p >\subset Isom^+(\mathbb{H}^n)$ with h hyperbolic and p parabolic, one fixes an horoball $\mathcal{H}_0$ centered at the fixed point of p, hence a neighborhood $< p > \backslash \mathcal{H}_0$ of the cusp associated with p. Then, one chooses a profile τ which does not modify the critical exponent $\delta_{\langle p \rangle}$ of $\langle p \rangle$ and makes the group Γ a convergent lattice of a new manifold X. It can then be shown that there exists a critical value a^* such that

(1) for any $a > a^*$ the metric perturbed by τ_a in the cusp (more explicitely the metric whose profile is τ beyond the height a - see again Section 2.2) makes Γ non exotic, hence divergent,

(2) the one given by τ_{a^*} makes Γ exotic and divergent.

In the first case, the divergence comes from the contribution of elements of $\Gamma_c \subset \Gamma$ corresponding to geodesic loops staying at height less than a in the cusp, which is preponderant in the value of the Poincaré series of Γ since $\delta_{\langle p \rangle}$ is strictly less than the critical exponent of $\Gamma \subset Isom^+(\mathbb{H}^n)$.

Here, we adapt this approach to obtain a discrete group Γ with finite covolume, in dimension 2. We start from the surface $\bar{X} = \Gamma\backslash X$ with $r + 1$ cusps described in 4.2, with a dominant cusp $\bar{\mathcal{C}}_r = P_r\backslash\mathcal{H}_r$ and make Γ convergent by choosing $a_0, ..., a_r \gg 0$, as in Theorem 1.2. Besides a different coding by blocks (due to the generators which are all parabolic) which gives a slightly different expression for the transfer operator associated to Γ, the main difficulty here is to show that Γ can be made divergent. This cannot be achieved now by simply pushing the perturbation far away in the cusp, since in our case δ_{P_r} is strictly greater than the critical exponent of the subset Γ_a of elements staying in the compact, non-perturbed part ; so, even choosing the $a_i \gg 0$, the group Γ remains convergent ! To obtain the divergence we rather modify $\bar{\mathcal{C}}_r$ with a profile $\tau_{r,a_r,b}$ which equals the profile of a cusp with constant curvature metric $-\omega_{\tau_r}^2$ on a sufficiently large band of width $b \gg 0$. With those conventions, we will prove the existence of b^* such that :

(1) Γ with the metric perturbed by $\tau_{a,b}$ is non exotic and divergent for every $b > b^*$,

(2) Γ with the metric perturbed by τ_{a,b^*} is exotic and still divergent.

The approach to obtain the existence of the critical value b^* is to consider a family of transfer operators $(\mathcal{L}_{b,z})_{b,z}$ associated with the transformation T, the latter being described at the end of Section 4 and depending on b. The continuity of $b \mapsto \mathcal{L}_{b,z}$ and $b \mapsto \rho(\mathcal{L}_{b,z})$ where $\rho(\mathcal{L}_{b,z})$ is the spectral radius of $\mathcal{L}_{b,z}$ in some suitable functional Banach space is a key point of the approach.

5.1. On the spectrum of transfer operators. The map T encodes a large part of the action of the group Γ on $\partial_0 X$; from an analytic point of view, this dynamic may be described throughout the family $(\mathcal{L}_{b,z})_{b,z}$ of *transfer operators* associated with T, which takes into account the different inverse branches of T and is defined formally by : for any Borel bounded function $\varphi : \partial X \to \mathbb{R}$ and any $x \in \partial_0 X$

$$\mathcal{L}_{b,z}\varphi(x) = \sum_{y/Ty=x} e^{-z\mathfrak{C}(y)}\varphi(y), \tag{9}$$

for a "ceiling" function $\mathfrak{C}$ to be defined. The function $\mathfrak{C}$ will depend on the metric $g_{a_0,\cdots,a_r,b}$ and especially on the width b of the band inside the cusp $\bar{\mathcal{C}}_r$ where the curvature is prescribed to be $-\omega_\tau^2$; we will need to understand precisely the dependence of $\mathcal{L}_{z,b}$ with respect to b and thus notice the ceiling function $\mathfrak{C}_b$ and its corresponding transfer operator $\mathcal{L}_{b,z}$.

The alphabet $\mathfrak{B}$ being countable, the pre-images of $x \in \partial_0 X$ by T are the points $y = \beta \cdot x$, for those blocks $\beta \in \mathfrak{B}$ such that x belongs to K_β, that is :

- if $x \in I'_i$ then $\beta \in \cup_{j\neq i}\widehat{\mathcal{P}}_j$ or $\beta \in \mathcal{W}_1$ with $l_\beta \neq i$,
- if $x \in I_i \setminus I'_i$ then $\beta \in \cup_{j\neq i}\widehat{\mathcal{P}}_j$.

For such $y = \beta \cdot x$, the quantity $\mathfrak{C}_b(y)$ is given by

$$\mathfrak{C}_b(y) := \mathcal{B}_{\beta^{-1}\cdot y}(\beta^{-1}\cdot \mathbf{x}_0, \mathbf{x}_0) = \mathcal{B}_x(\beta^{-1}\cdot \mathbf{x}_0, \mathbf{x}_0) \tag{10}$$

(where $\mathcal{B}$ is the Busemann function with respect to the metric $g = g_{a_0,\cdots,a_r,b}$). By Lemma 4.3, the quantity $\mathcal{B}_x(\beta^{-1}\cdot \mathbf{x}_0, \mathbf{x}_0)$ differs from $d(\beta^{-1}\cdot \mathbf{x}_0, \mathbf{x}_0)$ by a term which is bounded uniformly in $\beta \in \mathfrak{B}$ and $x \in K_\beta$, this will allow us to compare the sum $\sum_{k\geq 0} \mathcal{L}^k_{b,s}1(x)$ with the Poincaré series $P_\Gamma(\mathbf{x_0}, \mathbf{x_0}, s) = \sum_{\gamma\in\Gamma} e^{-sd(\mathbf{x}_0,\gamma\cdot\mathbf{x}_0)} \simeq \sum_{k\geq 0} \sum_{\substack{\gamma\in\Gamma\\ |\gamma|_{\mathfrak{B}}=k}} e^{-sd(\mathbf{x}_0,\gamma\cdot\mathbf{x}_0)}$.

We can now make explicit the definition of transfer operators $\mathfrak{L}_{b,z}$, extending formula (9) for any $x \in \partial X$: for $b \geq 0, z \in \mathbb{C}$, any bounded Borel function $\varphi : \partial X \to \mathbb{C}$ and any $x \in \partial X$ we set

$$\mathcal{L}_{b,z}\varphi(x) = \sum_{\beta\in\mathfrak{B}} 1_{K_\beta}(x)e^{-z\mathcal{B}_x(\beta^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)}\varphi(\beta\cdot x). \tag{11}$$

In other words, $\mathcal{L}_{b,z}\varphi(x) = \sum_{\beta\in\mathfrak{B}} w_{b,z}(\beta, x)\varphi(\beta\cdot x)$ where the $w_{b,z}(\gamma,\cdot) : \partial X \to \mathbb{C}, z \in \mathbb{C}$ and $\gamma \in \Gamma$ are defined by

$$w_{b,z}(\gamma, x) = 1_{K_\gamma}(x)e^{-z\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)}$$

and called *weight functions.* Observe now that these functions satisfy the cocycle relation : if the $\mathfrak{B}$-decomposition of $\gamma = \gamma_1\gamma_2$ is given by the simple concatenation $\gamma_1 * \gamma_2$ of the γ_i, i.e. $|\gamma_1\gamma_2|_{\mathfrak{B}} = |\gamma_1|_{\mathfrak{B}} + |\gamma_2|_{\mathfrak{B}}$, then

$$w_{b,z}(\gamma_1\gamma_2, x) = w_{b,z}(\gamma_1, \gamma_2\cdot x)\cdot w_{b,z}(\gamma_2, x).$$

This equality leads to the following simple expression of the iterates of the transfer operators : for any $k \geq 1$ and $x \in \partial X$

$$\mathcal{L}^k_{b,z}\varphi(x) = \sum_{\substack{\gamma\in\Gamma\\ |\gamma|_{\mathfrak{B}}=k}} w_{b,z}(\gamma, x)\varphi(\gamma\cdot x) = \sum_{\substack{\gamma\in\Gamma\\ |\gamma|_{\mathfrak{B}}=k}} 1_{K_\gamma}(x)e^{-z\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)}\varphi(\gamma\cdot x). \tag{12}$$

The operator $\mathcal{L}_{b,z}$ is well defined when $\mathrm{Re}(z) > \delta$, and also for $\mathrm{Re}(z) = \delta$ if Γ is convergent. It acts on the space $C(\partial X)$ of $\mathbb{C}$-valued continuous functions on ∂X endowed with the norm $|\cdot|_\infty$ of the uniform convergence; however, to obtain a quasi-compact operator with good spectral properties, we will consider its restriction to a subspace $\mathbb{L}_\alpha \subset \mathrm{C}(\partial X)$ of Hölder continuous functions with respect to D_0, for α given by Lemma 4.2. Namely we let

$$\mathbb{L}_\alpha := \{\varphi \in C(\partial X) : \|\varphi\| = |\varphi|_\infty + [\varphi]_\alpha < +\infty\}$$

where $[\varphi]_\alpha := \sup_{\substack{x,y\in\partial X\\ x\neq y}} \dfrac{|\varphi(x)-\varphi(y)|}{D_0^\alpha(x,y)}$; then, $\mathcal{L}_{b,z}$ acts on $\mathbb{L}_\alpha$ because of the following

Lemma 5.1. *Each weight $w_{b,z}(\gamma,\cdot)$ belongs to $\mathbb{L}_\alpha$ and for any $z \in \mathbb{C}$, there exists $C = C(z) > 0$ such that for any $\gamma \in \Gamma$*

$$\|w_{b,z}(\gamma,\cdot)\| \leq Ce^{-\mathrm{Re}(z)d_b(\mathbf{x}_0,\gamma\cdot\mathbf{x}_0)}.$$

Proof. By Lemma 4.3, the family $\{e^{\Re(z)d_b(\mathbf{o},\gamma.\mathbf{o})}|w_{b,z}(\gamma,.)|_\infty,\ \gamma \in \Gamma\}$ is bounded. The control of the Lipschitz-coefficient of $w_{b,z}(\gamma,.)$ is more delicate, we first recall briefly the proof in the constant curvature case and refer Lemma III.3 in [**3**] in variable curvature. To estimate $w_{b,z}(\gamma,x) - w_{b,z}(\gamma,y)$ for any points x,y belonging to the same subset K_γ, note that there exists a constant $A > 0$ such that $|\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0) - \mathcal{B}_y(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)| \leq A|x-y|$. The inequality $|e^Z - 1| \leq 2|Z|e^{|\Re(Z)|}$ readily implies

$$|e^{-z\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)} - e^{-z\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)}| \leq 2A|z|e^{A|z|\times|x-y|}e^{-\Re(z)\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)}|x-y|.$$

So, the Hölder coefficient of $w_z(\gamma,.)$ satisfies $[w_z(\gamma,.)]_\alpha \leq Ce^{-\Re(z)d(\mathbf{o},\gamma.\mathbf{o})}$ for some constant $C = C(z)$. □

The following theorem plays a key rule in the sequel ; it allows us to control the spectrum of the operators $\mathcal{L}_{b,s}$ for real parameters $b \geq 0$ and $s \geq \delta := \omega_\tau/2$.

Theorem 5.2. *For any $b \geq 0$ and $s \geq \delta = \omega_\tau/2$, the operator $\mathcal{L}_{b,s}$ acts both on $(C(\partial X), |\cdot|_\infty)$ and $(\mathbb{L}_\alpha, \|\cdot\|)$, with respective spectral radius $\rho_\infty(\mathcal{L}_{b,s})$ and $\rho_\alpha(\mathcal{L}_{b,s})$. Furthermore, the operator $\mathcal{L}_{b,s}$ is quasi-compact*[3] *on $\mathbb{L}_\alpha$, and :*

(1)
- $\rho_\alpha(\mathcal{L}_{b,s}) = \rho_\infty(\mathcal{L}_{b,s})$,
- *$\rho_\alpha(\mathcal{L}_{b,s})$ is a simple, isolated eigenvalue of $\mathcal{L}_{b,s}$,*
- *the eigenfunction $h_{b,s}$ associated with $\rho_\alpha(\mathcal{L}_{b,s})$ is non negative on ∂X,*

(2) For any $s \geq \delta$, the map $b \mapsto \mathcal{L}_{b,s}$ is continuous from $\mathbb{R}^+$ to the space of continuous linear operators on $\mathbb{L}_\alpha$,

(3) The map $s \mapsto \rho_\infty(\mathcal{L}_{b,s})$ is decreasing on $[\delta, +\infty[$.

Sketch of the proof. We follow Sections 4.3 and 4.4 in [**30**]. The key argument is the following inequality : for any $\beta \in \mathfrak{B}, x, y \in K_\beta, s \geq \delta$ and $k \geq 1$

$$\begin{aligned}|\mathcal{L}^k_{b,s}\varphi(x) - \mathcal{L}^k_{b,s}\varphi(y)| \quad &\leq \quad \sum_{\substack{\gamma\in\Gamma \\ |\gamma|_{\mathfrak{B}}=k}} w_{b,s}(\gamma,x)|\varphi(\gamma\cdot x) - \varphi(\gamma\cdot y)| \\ &\qquad + \sum_{\substack{\gamma\in\Gamma \\ |\gamma|_{\mathfrak{B}}=k}} |w_{b,s}(\gamma,x) - w_{b,s}(\gamma,y)| \times |\varphi|_\infty,\end{aligned}$$

so that, by Corollary 4.4, there exists sequences $(r_k)_k$ and $(R_k)_k$ such that

$$\|\mathcal{L}^k_{b,s}\varphi\| \leq r_k\|\varphi\| + R_k|\varphi|_\infty \tag{13}$$

with $\limsup_{k\to+\infty} r_k^{1/k} \leq r|\mathcal{L}_{b,s}|_\infty$. Using a version due to H. Hennion of the Ionescu-Tulcea-Marinescu's theorem concerning quasi-compact operators, we conclude that the essential spectral radius of $\mathcal{L}_{b,s}$ on $\mathbb{L}_\alpha$ is less than $r\rho_\infty(\mathcal{L}_{b,s})$. The control of the peripheral spectrum of $\mathcal{L}_{b,s}$ in $\mathbb{L}_\alpha$ is based on the positivity of this family of operators, as in [**30**] ; similarly, one adapts the proof of Proposition 4.7 in [**30**] to establish assertion (2). Proof of (3) is direct.□

Remark. 5.3. Let $\rho_{b,s} = \rho_\infty(\mathcal{L}_{b,s})$. Then, $\mathcal{L}_{b,s}h_{b,s} = \rho_{b,s}h_{b,s}$, the function $h_{b,s}$ being unique up to a multiplicative constant. By duality, there also exists a unique probability measure $\sigma_{b,s}$ on ∂X such that $\sigma_{b,s}\mathcal{L}_{b,s} = \rho_{b,s}\sigma_{b,s}$; the function $h_{b,s}$ becomes uniquely determined imposing the condition $\sigma_{b,s}(h_{b,s}) = 1$, which we will assume from now on.

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3. In other words its essential spectral radius on this space is less than $\rho_\alpha(\mathcal{L}_{b,s})$

5.2. From convergence to divergence : Proof of Theorem 1.3.

Combining expression (12) with Lemma 4.3, one gets for any $s, b > 0$ and $k \geq 0$

$$|\mathcal{L}_{b,s}^k 1|_\infty \asymp \sum_{\substack{\gamma\in\Gamma \\ |\gamma|_{\mathfrak{B}}=k}} \exp(-s d_b(\mathbf{x}_0, \gamma\cdot\mathbf{x}_0)). \tag{14}$$

Consequently, the Poincaré series $P_\Gamma(s)$ of Γ relatively to d_b and the series $\sum_{k\geq 0} |\mathcal{L}_{b,s}^k 1|_\infty$ converge or diverge simultaneously. Following [**30**], we see that the function $s \mapsto \rho_\infty(\mathcal{L}_{b,s})$ is strictly decreasing on $\mathbb{R}^+$; the Poincaré exponent of Γ relatively to d_b is then equal to

$$\delta_\Gamma = \sup\Big\{s \geq 0 : \rho_\infty(\mathcal{L}_{b,s}) \geq 1\Big\} = \inf\Big\{s \geq 0 : \rho_\infty(\mathcal{L}_{b,s}) \leq 1\Big\}.$$

and the latter expression will be useful to prove Theorem 1.3. We first get the

LEMMA 5.4. *Assume that the profiles $\tau_0, \cdots, \tau_1, \cdots, \tau_r = \tau$ are convergent and satisfy the condition $\omega_\tau = \max(\omega_{\tau_0}, \cdots, \omega_{\tau_r}) > 2$. Then there exist non negative reals $a_0, \cdots, a_r$ and $b_0 > 0$ such that*

- *The group Γ has exponent $\frac{\omega_\tau}{2}$ and is convergent with respect to $g_{a_0,\cdots,a_r,0}$;*
- *The group Γ has exponent greater than $\frac{\omega_\tau}{2}$ and is divergent with respect to $g_{a_0,\cdots,a_r,b}$ for any $b \geq b_0$.*

Proof. By the previous Section and the choice of the profiles $\tau_i, 0 \leq i \leq r$ (with the additionnal notation $\tau = \tau_r$), we may fix the constants $a_0, \cdots, a_r$ large enough, in order that the group Γ acting on $(X, g_{a_0,\cdots,a_r,0})$ is a convergent lattice with exponent $\delta = \frac{\omega_\tau}{2}$. To prove the second point we will take in account only the contribution of words containing powers of p_r ; we get by the triangle inequality

$$\begin{aligned}
\sum_{\gamma\in\Gamma} e^{-\delta d_b(\mathbf{x}_0,\gamma\cdot\mathbf{x}_0)} &\geq \sum_{k\geq 1} \sum_{\substack{Q_1,\cdots,Q_k\in\mathcal{W}_1 \\ |l_1|,\cdots,|l_k|\geq 2}} e^{-\delta d_b(\mathbf{x}_0, p_r^{l_1}Q_1\cdots p_r^{l_k}Q_k\cdot\mathbf{x}_0)} \\
&\geq \sum_{k\geq 1}\left(\sum_{|l|\geq 2} e^{-\delta d_b(\mathbf{x}_0,p_r^l\cdot\mathbf{x}_0)} \sum_{Q\in\mathcal{W}_1} e^{-\delta d_b(\mathbf{x}_0,Q.\mathbf{x}_0)}\right)^k.
\end{aligned}$$

We now use the following fact :

LEMMA 5.5. *There exist $b_0 > 0$ and, for any $b \geq b_0$, integers $n_{a_r} \leq n_b$ depending respectively on a_r and $b \geq b_0$, such that :*

- *$n_b \to \infty$ when $b \to \infty$,*
- *for any l satisfying $n_{a_r} \leq |l| \leq n_b$, there exists a constant C depending only on a_r and the bounds of the curvature such that :*

$$d_b(\mathbf{x}_0, p^l\cdot\mathbf{x}_0) \geq \frac{2}{\omega_\tau}\ln|l| - C.$$

Proof. When $|l|$ is large enough, say $|l| \geq n_{a_r} \geq 2$, the geodesic segments $[\mathbf{x}_0, p^l\cdot\mathbf{x}_0]$ intersect the horoball $\mathcal{H}_{\xi_r}(a_r + \Delta)$ and when $|l|$ is not too big the same geodesics do not intersect $\mathcal{H}_{\xi_r}(a_r+b+\Delta)$. If we set $n_b := \max\{n \in \mathbb{N} \ ; \ |k| \leq n \Rightarrow [\mathbf{x}_0, p^k\mathbf{x}_0]\cap\mathcal{H}_{\xi_r}(a_r+b+\Delta) = \emptyset\}$, the latter is well defined and satisfies $n_b \to \infty$ when $b \to \infty$. Fix l satisfying $n_{a_r} \leq |l| \leq n_b$ and define respectively by $\mathbf{x}_1$ and $\mathbf{x}_2$ the enter and exit point of the oriented geodesic segment $[\mathbf{x}_0, p^l\cdot\mathbf{x}_0]$ in $\mathcal{H}_{\xi_r}(a_r+\Delta)$. Using a comparison argument with the geodesic from $\mathbf{x}_0$ tangent to $\partial\mathcal{H}_{\xi_r}(a_r+\Delta)$, we can observe that

$$|d_b(\mathbf{x}_0, \mathbf{x}_1) - d_0(\mathbf{x}_0, \mathbf{x}_1)| < c \quad \text{and} \quad |d_0(\mathbf{x}_0, \mathbf{x}_1) - a_r| < c$$

for a constant c only depending on the bounds of the curvature. In the horospherical annulus $\mathcal{H}_{\xi_r}(a_r+\Delta)\backslash\mathcal{H}_{\xi_r}(a_r+b+\Delta)$ the curvature is $-\omega_\tau^2$ so that $d_b(\mathbf{x}_1, \mathbf{x}_2) = \frac{d_{hyp}(\mathbf{x}_1,\mathbf{x}_2)}{\omega_\tau}$. Combining this with the above inequalities we get the existence of a constant C depending on a_r, τ and on the bounds of the curvature such that for l satisfying $n_{a_r} \leq |l| \leq n_b$:

$$d_b(\mathbf{x}_0, p_r^l\mathbf{x}_0) \geq \frac{d_{hyp}(\mathbf{x}_0, p^l\mathbf{x}_0)}{\omega_\tau} - C = \frac{2}{\omega_\tau}\ln(|l|) - C. \qquad \square$$

To conclude the proof of Lemma 5.4, notice that for $\delta = \frac{\omega_\tau}{2}$ we have :

$$\sum_{\{l\ ;\ |l|\geq 2\}} e^{-\delta d_b(\mathbf{x}_0, p^l\cdot\mathbf{x}_0)} \geq \sum_{|l|=n_{a_r}}^{n_b} e^{-\delta d_b(\mathbf{x}_0, p^l\cdot\mathbf{x}_0)} \succeq \sum_{|l|=n_{a_r}}^{n_b} \left(\frac{1}{|l|^{2/\omega_\tau}}\right)^{\omega_\tau/2} \to +\infty \quad \text{as} \quad b\to+\infty.$$

So, there exists $b_0 \geq 0$ such that $\left(\sum_{|l|\geq 2} e^{-\delta d_b(\mathbf{x}_0, p^l\cdot\mathbf{x}_0)} \sum_{Q\in\mathcal{W}_1} e^{-\delta d_b(\mathbf{x}_0, Q\cdot\mathbf{x}_0)}\right) > 1$ as soon as $b \geq b_0$; in particular, continuity and monotonicity in s gives :

$$\left(\sum_{|l|\geq 2} e^{-sd_{b_0}(\mathbf{x}_0, p^l\cdot\mathbf{x}_0)} \sum_{Q\in\mathcal{W}_1} e^{-sd_{b_0}(\mathbf{x}_0, Q\cdot\mathbf{x}_0)}\right) > 1$$

for some $s > \frac{\omega_\tau}{2}$. This ensures that $\delta_\Gamma \geq s > \frac{\omega_\tau}{2} = \delta_{\mathcal{P}_r} = \max\{\delta_{\mathcal{P}_i} \,|\, 0\leq i\leq r\}$, and Γ is divergent with respect to the metric $g_{a_0,\cdots,a_r,b}$ by the critical gap property recalled in the introduction. $\square$

End of proof of Theorem 1.3. Recall that $\delta = \omega_\tau/2$. Since $\rho_\alpha(\mathcal{L}_{b,\delta})$ is an eigenvalue of $\mathcal{L}_{b,\delta}$ which is isolated in the spectrum of $\mathcal{L}_{b,\delta}$, the function $b\mapsto \rho_\alpha(\mathcal{L}_{b,\delta})$ (which is *a priori* semi-continuous) has the same regularity as $b\mapsto \mathcal{L}_{b,\delta}$. For b_0 given by Lemma 5.4, we have $\rho_\alpha(\mathcal{L}_{0,\delta}) = \rho_\infty(\mathcal{L}_{0,\delta}) \leq 1$ and $\rho_\alpha(\mathcal{L}_{b_0,\delta}) = \rho_\infty(\mathcal{L}_{b_0,\delta}) \geq 1$; thus, there exists $b_* \in [0, b_0]$ such that $\rho_\alpha(\mathcal{L}_{b^*,\delta}) = \rho_\infty(\mathcal{L}_{b^*,\delta}) = 1$. Since the function $s\mapsto \rho_\infty(\mathcal{L}_{b^*,s})$ is strictly decreasing on $[\delta,+\infty[$, one gets $\rho_\infty(\mathcal{L}_{b^*,s}) < 1$ as soon as $s > \delta$. For such values of s, the Poincaré series $P_\Gamma(s)$ of Γ relatively to the metric g_{b^*} thus converges, this implies that its Poincaré exponent δ_{Γ,b^*} is not greater than δ ; actually we have $\delta_{\Gamma,b^*} = \delta$ since $\delta_{\langle p_r\rangle} = \delta$ and $p_r \in \Gamma$. Finally, the eigenfunction $h_{b^*,\delta}$ of $\mathcal{L}_{b^*,\delta}$ associated with $\rho_\alpha(\mathcal{L}_{b^*,\delta})$ being non negative on ∂X, one gets $h_{b^*,\delta} \asymp 1$ and so

$$\sum_{k\geq 0} |\mathcal{L}^k_{b^*,\delta} 1|_\infty \asymp \sum_{k\geq 0} |\mathcal{L}^k_{b^*,\delta} h_{b^*,\delta}|_\infty = \sum_{k\geq 0} |h_{b^*,\delta}|_\infty = \infty$$

which implies, by (14), that Γ is divergent with respect to the metric g_{b^*}. $\square$

6. Counting for some divergent exotic lattices

We prove here the following general result, which implies Theorem 1.4 given in the Introduction :

THEOREM 6.1. *Let $\bar{X}$ be a $(r+1)$-punctured sphere, endowed with a metric of finite area which has a cusp $\bar{\mathcal{C}}_i$ with profile τ_i for each puncture and is hyperbolic outside $\bar{\mathcal{C}}_0\cup\bar{\mathcal{C}}_1\cup\cdots\bar{\mathcal{C}}_r$. Let $p_0,\cdots,p_r$ be parabolic isometries, each one generating a parabolic subgroup $\mathcal{P}_i$ associated to $\bar{\mathcal{C}}_i$, such that $\Gamma = \pi_1(\bar{X})$ is a free group over $p_1,\cdots,p_r$ and $p_0 = p_1\cdots p_r$. Let X be the universal cover of $\bar{X}$, let $\mathbf{x}_0\in X$, and assume that the lifted metric g on X satisfies the following assumptions :*

- $\mathbf{H_1}$ *: the group Γ is exotic and divergent with Poincaré exponent $\delta = \delta_\Gamma = \frac{\omega}{2}$, where $\omega = \omega_{\tau_r} = \max(\omega_{\tau_0},\cdots,\omega_{\tau_r}) > 2$;*
- $\mathbf{H_2}$ *: there exist $\kappa \in]1/2, 1[$ and a slowly varying function* [4] *L such that*

$$\sum_{p\in\mathcal{P}_r/d(\mathbf{x}_0,p\cdot\mathbf{x}_0)>t} e^{-\delta d(\mathbf{x}_0,p\cdot\mathbf{o})} \overset{t\to+\infty}{\sim} \frac{L(t)}{t^\kappa}. \tag{15}$$

- $\mathbf{H_3}$ *: the parabolic groups $\mathcal{P}_l$, for $0\leq l\leq r-1$, satisfy the condition*

$$\sum_{p\in\mathcal{P}_l/d(\mathbf{x}_0,p\cdot\mathbf{x}_0)>t} e^{-\delta d(\mathbf{x}_0,\gamma\cdot\mathbf{x}_0)} = o\Big(\frac{L(t)}{t^\kappa}\Big). \tag{16}$$

4. A function $L(t)$ is said to be "slowly varying" or "of slow growth" if it is positive, measurable and $L(\lambda t)/L(t)\to 1$ as $t\to+\infty$ for every $\lambda > 0$.

Then, for any $1 \le j \le r$ and any fixed $x_j \in \partial X$ far enough from the fixed point of p_j, there exists $C_j > 0$ such that

$$\sharp\{\gamma \in \Gamma_j / \mathcal{B}_{x_j}(\gamma^{-1}\cdot \mathbf{x}_0, \mathbf{x}_0) \le R\} \overset{R\to+\infty}{\sim} C_j \frac{e^{\delta_\Gamma R}}{R^{1-\kappa} L(R)}. \tag{17}$$

where Γ_j is the set of $\gamma \in \Gamma$ with last letter j, with respect to the alphabet $\mathcal{A}$.
As a consequence, we have

$$\sharp\{\gamma \in \Gamma / d(\mathbf{x}_0, \gamma\cdot \mathbf{x}_0) \le R\} \overset{R\to+\infty}{\asymp} \frac{e^{\delta_\Gamma R}}{R^{1-\kappa} L(R)}. \tag{18}$$

Notice that (18) easily follows from (17) summing over $j \in \{1, \cdots, r\}$, as for each j there exists $c = c(x_j) > 0$ such that $d(\mathbf{x}_0, \gamma\cdot\mathbf{x}_0) - c \le \mathcal{B}_{x_j}(\gamma^{-1}\cdot\mathbf{x}_0, \mathbf{x}_0) \le d(\mathbf{x}_0, \gamma\cdot\mathbf{x}_0)$.

Let us make some remarks :

(1) We have seen in the previous Sections how to perturb a hyperbolic metric to obtain a negatively curved metric $g = g_{a_0,\cdots,a_r,b^*}$ on $\bar{X}$ with profiles $\tau_0, \cdots, \tau_r$ and parameters $a_0, \cdots, a_r$ and b^* so that the hypothesis $\mathbf{H_1}$ holds.

(2) Hypothesis $\mathbf{H_2}$ is inspired by probability theory and it corresponds to a *heavy tail condition* satisfied by random walks, which have been intensively investigated [**21**]. It holds in particular when

$$d(\mathbf{x}_0, p_r^n \cdot \mathbf{x}_0) = \frac{2\ln n + 2(1+\kappa)(\ln\ln n + O(n))}{\omega}$$

for some $\omega > 2$. This equality concerns only the asymptotic geometry on the cusp $\bar{\mathcal{C}}_r$ as it is equivalent to prescribe a profile without modifying its exponential growth rate. Hence it is compatible with any choice of the parameter b. The critical exponent of $\mathcal{P}_r$ is thus $\delta = \omega/2$ and one gets, as $t \to +\infty$,

$$\begin{aligned}
\sum_{p\in\mathcal{P}_r/d(\mathbf{x}_0,p\cdot\mathbf{x}_0)>t} e^{-\delta d(\mathbf{x}_0,p\cdot\mathbf{o})} &= \sum_{n\in\mathbb{N}/d(\mathbf{x}_0,p_r^n\cdot\mathbf{x}_0)>t} e^{-\delta d(\mathbf{x}_0,p_r^n\cdot\mathbf{o})} \\
&\asymp \sum_{n>\frac{e^{\omega t/2}}{t^{1+\kappa}}} \frac{1}{n(\ln n)^{1+\kappa}} \\
&\asymp \int_{\frac{e^{\omega t/2}}{t^{1+\kappa}}}^{+\infty} \frac{du}{u(\ln u)^{1+\kappa}} \\
&\asymp \frac{1}{t^\kappa}.
\end{aligned}$$

(3) The condition $\kappa \in]1/2, 1[$ readily implies

$$\begin{aligned}
\sum_{p\in\mathcal{P}_r} d(\mathbf{x}_0, p\cdot\mathbf{x}_0) e^{-\delta d(\mathbf{x}_0,p\cdot\mathbf{x}_0)} &\asymp \sum_{N\ge1} N\Big(\sum_{p\in\mathcal{P}_r/N<d(\mathbf{x}_0,p\cdot\mathbf{x}_0)\le N+1} e^{-\delta d(\mathbf{x}_0,p\cdot\mathbf{x}_0)}\Big) \\
&\asymp \sum_{N\ge1}\Big(\sum_{p\in\mathcal{P}_r/d(\mathbf{x}_0,p\cdot\mathbf{x}_0)>N} e^{-\delta d(\mathbf{x}_0,p\cdot\mathbf{o})}\Big) \\
&\asymp \sum_{N\ge1} \frac{L(N)}{N^\kappa} = +\infty.
\end{aligned}$$

By Theorem B in [**13**], it readily follows that the Bowen-Margulis measure associated with Γ is infinite.

(4) Finally, notice that hypothesis $\mathbf{H_3}$ is satisfied in particular when the gap property $\delta_{\mathcal{P}_l} < \delta$ holds for any $0 \le l \le r-1$.

Under these assumptions, we will see that the subgroup $\mathcal{P}_r$ corresponding to the cusp $\mathcal{C}_r$ has a dominant influence on the behavior of the orbital function of Γ.

We present here the main steps of the proof of Theorem 6.1, and refer to [**19**] for details. Let $\bar{X} = \Gamma\backslash X$, where X is its universal cover. First notice that, under the assumption of the theorem, we can assume (by replacing the cusps $\bar{\mathcal{C}}_i$ with purely hyperbolic cusps) that Γ acts on the disk model of $\mathbb{H}^2$ with a fundamental polygon $\mathcal{D}$ as described at the beginning of Section 4, with $2r$ boundary points $\xi_0 = \eta_0, \xi_1, \eta_1, \cdots, \xi_r, \eta_r = \xi_0$, enumerated in cyclic order, with $p_i \cdot \xi_i = \xi_i$ and $p_i \cdot \eta_{i-1} = \eta_i$ for $1 \leq i \leq r$, and all points η_i belonging to the Γ-orbit of the fixed point ξ_0 of p_0 ; so, we can assume that the metric g is a modification of the hyperbolic one on the fundamental system of horospheres $\mathcal{H}_{\xi_i}$, with $\bar{\mathcal{C}}_i = P_i\backslash\mathcal{H}_{\xi_i}$. We therefore have a coding of elements of Γ by blocks $\beta \in \mathfrak{B}$ and a ping-pong dynamic, for the alphabet $\mathfrak{B}$ as described in Section 4.

For any $R \geq 0$, let us denote $W_j(R, \cdot)$ the measure on $\mathbb{R}$ defined by : for any Borel non negative function $\psi : \mathbb{R} \to \mathbb{R}$

$$W_j(R,\psi) := \sum_{\gamma\in\Gamma_j} e^{-\delta\mathcal{B}_{x_j}(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)}\psi(\mathcal{B}_{x_j}(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)-R).$$

One gets $0 \leq W_j(R,\psi) < +\infty$ when ψ has a compact support in $\mathbb{R}$ since the group Γ is discrete, furthermore $\sum_{j=1}^r W_j(R,\psi) = e^{-\delta R}v_\Gamma(R)$ when $\psi(t) = e^{\delta t}1_{t\leq 0}$. If one proves that for any non negative and continuous function ψ with compact support and such that $\int_{\mathbb{R}^+}\psi(x)dx > 0$

$$W_j(R,\psi) \quad \overset{R\to+\infty}{\sim} \quad \frac{C_j}{R^{1-\kappa}L(R)}\int_{\mathbb{R}}\psi(x)dx, \tag{19}$$

this convergence will also hold for non negative functions with compact support in $\mathbb{R}$ and whose discontinuity set has 0 Lebesgue measure ; Theorem 6.1 follows since $v_\Gamma(R) = e^{\delta R}\sum_{j=1}^r\sum_{n\geq 0} W_j(R, e^{\delta t}1_{]-(n+1),-n]}(t))$. From now on, we fix a continuous function $\psi : \mathbb{R} \to \mathbb{R}^+$ with compact support ; one gets, for $1 \leq j \leq r$ fixed

$$W_j(R,\psi) = \sum_{k\geq 0}\Big(\sum_{\gamma\in\Gamma_j/|\gamma|_{\mathfrak{B}}=k} e^{-\delta\mathcal{B}_{x_j}(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)}\psi(\mathcal{B}_{x_j}(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0)-R)\Big).$$

Notice that for $\gamma \in \Gamma_j$ with $\mathfrak{B}$-decomposition $\gamma = \beta_1\cdots\beta_k$, one gets , setting $y := \gamma \cdot x_j$,

$$\begin{aligned}\mathcal{B}_{x_j}(\gamma^{-1}\cdot\mathbf{x}_0,\mathbf{x}_0) &= \mathfrak{C}(\beta_1\cdots\beta_k\cdot x_j) + \mathfrak{C}(\beta_2\cdots\beta_k\cdot x_j) + \cdots + \mathfrak{C}(\beta_k\cdot x_j)\\ &= \mathfrak{C}(y) + \mathfrak{C}(T\cdot y) + \cdots + \mathfrak{C}(T^{k-1}\cdot y) = S_k\mathfrak{C}(y)\end{aligned}$$

where $\mathfrak{C}$ is the "ceiling " function defined as in (10), so that

$$W_j(R,\psi) = \sum_{k\geq 0}\sum_{y\in\partial X/T^k\cdot y=x_j} e^{-\delta S_k\mathfrak{C}(y)}\psi(S_k\mathfrak{r}(y)-R). \tag{20}$$

By a classical argument in probability theory due to Stone (see for instance [**21**]), it suffices to check that the convergence (19) holds when ψ has a C^∞ Fourier transform with compact support : indeed, the test function ψ may vary in the set $\mathcal{H}$ of functions of the form $\psi(x) = e^{itx}\psi_0(x)$ where ψ_0 is an integrable and strictly positive function on $\mathbb{R}$ whose Fourier transform is C^∞ with compact support. When $\psi \in \mathcal{H}$, one can use the inversion Fourier formula and write $\psi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\hat{\psi}(t)dt$, so that, for any $0 < s < 1$

$$\begin{aligned}W_j(s,R,\psi) &:= \sum_{k\geq 0}s^k\sum_{y\in\partial X/T^k\cdot y=x_j} e^{-\delta S_k\mathfrak{C}(y)}\psi(S_k\mathfrak{r}(y)-R)\\ &= \sum_{k\geq 0}s^k\sum_{y\in\partial X/T^k\cdot y=x_j}\frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}e^{-(\delta+it)S_k\mathfrak{C}(y)}\hat{\psi}(t)dt\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\hat{\psi}(t)\Big(\sum_{k\geq 0}s^k\sum_{y\in\partial X/T^k\cdot y=x_j} e^{-(\delta+it)S_k\mathfrak{C}(y)}\Big)dt\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\hat{\psi}(t)\Big(\sum_{k\geq 0}s^k\mathcal{L}^k_{\delta+it}1(x_j)\Big)dt\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\hat{\psi}(t)\,(I-s\mathcal{L}_{\delta+it})^{-1}\,1(x_j)dt \qquad (21)\end{aligned}$$

where $\mathcal{L}_z, z \in \mathbb{C}$, is the transfer operator associated to the function $\mathfrak{C}$ and the metric g, formally defined in Section 5.

When $\mathrm{Re}(z) \geq \delta$, we know that the $\mathcal{L}_z$ are bounded and quasi-compact on the space $\mathbb{L}_\omega(\partial X)$ of Hölder continuous function on $(\partial X, D_0)$. The subsection 6.1 is devoted to the control of the peripherical spectrum of $\mathcal{L}_{\delta+it}$ on $\mathbb{L}_\omega$. In subsection 6.2, we describe the local expansion of the dominant eigenvalue. Atlast we achieve the proof using arguments coming from renewal theory (subsection 6.3).

6.1. The spectrum of $\mathcal{L}_{\delta+it}$ on $\mathbb{L}_\omega$. First we need to control the spectrum of $\mathcal{L}_z$ when $z = \delta+it, t \in \mathbb{R}$. By Lemma 5.1, the operators $\mathcal{L}_z$ are bounded on $\mathbb{L}_\omega$ when $\mathrm{Re}(z) \geq \delta$; the spectral radius of $\mathcal{L}_z$ will be denoted $\rho_\omega(z)$ throughout this Section. In the following Proposition, we describe its spectrum on $\mathbb{L}_\omega$ when $\mathrm{Re}(z) = \delta$.

PROPOSITION 6.2. *There exist $\epsilon_0 > 0$ and $\rho_0 \in]0,1[$ such that, for any $t \in \mathbb{R}$ with modulus less than ϵ_0, the spectral radius $\rho_\omega(\delta+it)$ of $\mathcal{L}_{\delta+it}$ is $> \rho_0$ and the operator $\mathcal{L}_{\delta+it}$ has a unique eigenvalue λ_t of modulus $\rho(\delta+it)$, which is simple and closed to 1, the rest of the spectrum being included in a disc of radius ρ_0.*

Furthermore, for any $A > 0$, there exists $\rho_A \in]0,1[$ such that $\rho_\omega(\delta+it) \leq \rho_A$ for any $t \in \mathbb{R}$ such that $\epsilon_0 \leq |t| \leq A$.

NOTATION 6.3. *We denote σ the unique probability measure on ∂X such that $\sigma\mathcal{L}_\delta = \sigma$ and h the element of $\mathbb{L}_\omega$ such that $\mathcal{L}_\delta h = h$ and $\sigma(h) = 1$.*

Proof of Proposition 6.2 This is exactly the same proof that the one presented in [**2**](Proposition 2.2) and [**19**] : the operators $\mathcal{L}_{\delta+it}$ are quasi-compact on $\mathbb{L}_\omega$ and it is sufficient to control their peripherical spectrum. When t is closed to 0, we use the perturbation theory to conclude that the spectrum of $\mathcal{L}_{\delta+it}$ is closed to the one of $\mathcal{L}_\delta$: it is thus necessary to prove that the map $t \mapsto \mathcal{L}_{\delta+it}$ is continuous on $\mathbb{R}$. The following Lemma is devoted to precise the type of continuity of this function.

LEMMA 6.4. *Under the hypotheses $\mathbf{H_2}$ and $\mathbf{H_3}$, there exists a constant $C > 0$ such that*

$$\|\mathcal{L}_{\delta+it'} - \mathcal{L}_{\delta+it}\| \leq C|t'-t|^\kappa L\left(\frac{1}{|t'-t|}\right)$$

Proof. We will use the following classical fact ([**22**] p.272 and [**21**] Lemmas 1 & 2) :

LEMMA 6.5. *Let μ be a probability measure on $\mathbb{R}^+$, set $F_\mu(t) := \mu[0,t]$ and $m(t) := \int_0^t (1-F_\mu(s))ds$ and assume that there exist $\kappa \in \mathbb{R}$ and a slowly varying function L such that $1 - F_\mu(t) \sim \frac{L(t)}{t^\kappa}$ as $t \to +\infty$. One thus gets*

$$\lim_{t\to+\infty} \frac{t(1-F_\mu(t))}{m(t)} = 1-\kappa \tag{22}$$

and the characteristic function $\hat{\mu}(t) := \displaystyle\int_0^{+\infty} e^{itx}\mu(dx)$ of μ has the following local expansion as $t \to 0^+$

$$\hat{\mu}(t) = 1 - e^{-i\pi\frac{\kappa}{2}}\Gamma(1-\kappa)t^\kappa L\Big(\frac{1}{t}\Big)(1+o(t)). \tag{23}$$

Noticing that $\tilde{m}(t) := \int_0^t x\mu(dx) = m(t) - t(1-F_\mu(t))$ one also gets $\tilde{m}(t) \sim \kappa m(t)$ as $t \to +\infty$; furthermore, decomposing $\int_0^{+\infty} |e^{itx}-1|\ \mu(dx)$ as $\int_{[0,1/t]} |e^{itx}-1|\ \mu(dx) + \int_{]1/t,+\infty[} |e^{itx}-1|\ \mu(dx)$ and applying the previous estimations, one gets, for any $t \in \mathbb{R}$

$$\int_0^{+\infty} |e^{itx}-1|\ \mu(dx) \preceq t^\kappa L(1/t).$$

We now apply (22) with the probability measures $\mu_l, 1 \leq l \leq r$, on $\mathbb{R}^+$ defined by $\mu_l := c_l \sum_{p\in\mathcal{P}_l} \delta_{d(\mathbf{x}_0, p\cdot\mathbf{x}_0)}$ where $c_l > 0$ is some normalizing constant. As a direct consequence, under hypotheses $\mathbf{H_2}$ and $\mathbf{H_3}$, one gets (up to a modification of L by multiplicative constant)

$$\sum_{p\in\mathcal{P}_r/d(\mathbf{x}_0,p\cdot\mathbf{x}_0)\leq t} d(\mathbf{x}_0, p\cdot\mathbf{x}_0)e^{-\delta d(\mathbf{x}_0,p\cdot\mathbf{x}_0)} \overset{t\to+\infty}{\sim} \frac{t^{1-\kappa}}{1-\kappa}L(t). \tag{24}$$

and

$$\sum_{p\in\mathcal{P}_r}|e^{itd(\mathbf{x}_0,p\cdot\mathbf{x}_0)}-1|\times e^{-\delta d(\mathbf{x}_0,p\cdot\mathbf{x}_0)}\preceq t^\kappa L(1/t); \tag{25}$$

similarly, for $1\leq l<r$, one has

$$\sum_{p\in\mathcal{P}_l}|e^{itd(\mathbf{x}_0,p\cdot\mathbf{x}_0)}-1|\times e^{-\delta d(\mathbf{x}_0,p\cdot\mathbf{x}_0)}=t^\kappa L(1/t)o(t). \tag{26}$$

In the same way, by lemma 4.3, for any $x\in\partial X\setminus I_r$, one gets

$$\sum_{p\in\mathcal{P}_r/\mathfrak{C}(p\cdot x)\leq t}e^{-\delta\mathfrak{C}(p\cdot x))}\asymp\frac{L(t)}{t^\kappa},\qquad \sum_{p\in\mathcal{P}_r/\mathfrak{C}(p\cdot x))>t}\mathfrak{C}(p\cdot x)e^{-\delta\mathfrak{C}(p\cdot x)}\asymp t^{1-\kappa}L(t), \tag{27}$$

$$\sum_{p\in\mathcal{P}_r}|e^{it\mathfrak{C}(p\cdot x)}-1|\times e^{-\delta\mathfrak{C}(p\cdot x)}\preceq t^\kappa L(1/t) \tag{28}$$

and, for $1\leq l\leq r-1$ and $x\in\partial X\setminus I_l$

$$\sum_{p\in\mathcal{P}_l}|e^{it\mathfrak{C}(p\cdot x)}-1|\times e^{-\delta\mathfrak{C}(p\cdot x)}=\ t^\kappa L(1/t)o(t). \tag{29}$$

Notice now that the Poincaré series of the subset $\mathcal{W}_1\cdot\mathbf{x}_0$ is divergent, by sub-additivity of the orbital function $v_{W_1}(\mathbf{x}_0,R)=\{\beta\in\mathcal{W}_1/d(\mathbf{x}_0,\beta\cdot\mathbf{x}_0)\leq R\}$; we then deduce (by taking a free product with an element in Schottky position with $\mathcal{W}_1$, cp. [**16**]) that its exponential growth $\delta_{\mathcal{W}_1}=\limsup_{R\to+\infty}R^{-1}v_{W_1}(\mathbf{x}_0,R)$ is strictly less than δ. Therefore we also have, for any $1\leq l\leq r$ and $x\in\cup_{l'\neq l}I'_{l'}$

$$\sum_{\substack{\beta\in\mathcal{W}_1\\ l_\beta=l}}|e^{it\mathfrak{C}(\beta\cdot x)}-1|\times e^{-\delta\mathfrak{C}(\beta\cdot x)}\ t^\kappa L(1/t)o(t). \tag{30}$$

Noticing that for any $\beta\in\mathfrak{B}$ and $x\in K_\beta$

$$|w_{\delta+it}(\beta,x)-w_{\delta+it'}(\beta,x)|\leq|e^{i(t-t')\cdot x))}-1|\times e^{-\delta\mathfrak{C}(\beta\cdot x)}$$

one readily gets, combining the above estimations (27) to (30) all together

$$\left|\mathcal{L}_{\delta+it}-\mathcal{L}_{\delta+it'}\right|_\infty\leq\sum_{\beta\in\mathfrak{B}}|w_{\delta+it}(\beta,\cdot)-w_{\delta+it'}(\beta,\cdot)|_\infty\ \preceq|t-t'|^\kappa L\left(\frac{1}{|t-t'|}\right). \tag{31}$$

Now, for any $\beta\in\mathfrak{B}$ and $x,y\in K_\beta$, one gets

$$\begin{aligned}
&\Big|\big(w_{\delta+it}(\beta,x)-w_{\delta+it'}(\beta,x)\big)-\big(w_{\delta+it}(\beta,y)-w_{\delta+it'}(\beta,y)\big)\Big|\\
&\leq\ e^{-\delta\mathfrak{C}(\beta\cdot x)}\times\Big|\Big(e^{i(t-t')\mathfrak{C}(\beta\cdot x)}-1\Big)-\Big(e^{i(t-t')\mathfrak{C}(\beta\cdot y)}-1\Big)\Big|\\
&\qquad+\ \Big|e^{-\delta\mathfrak{C}(\beta\cdot x)}-e^{-\delta\mathfrak{C}(\beta\cdot y)}\Big|\times\Big|e^{i(t-t')\mathfrak{C}(\beta\cdot y)}-1\Big|\\
&\leq\ e^{-\delta\mathfrak{C}(\beta\cdot x)}\times\Big|e^{i(t-t')(\mathfrak{C}(\beta\cdot x)-\mathfrak{C}(\beta\cdot y))}-1\Big|\\
&\qquad+e^{-\delta\mathfrak{C}(\beta\cdot y)}\Big|e^{\delta(\mathfrak{C}(\beta\cdot y)-\mathfrak{C}(\beta\cdot x))}-1\Big|\times\Big|e^{i(t-t')\mathfrak{C}(\beta\cdot y)}-1\Big|\\
&\preceq\ \Big(e^{-\delta\mathfrak{C}(\beta\cdot x)}[r\circ\beta]\times|t-t'|+\ e^{-\delta\mathfrak{C}(\beta\cdot y)}[r\circ\beta]\times\Big|e^{i(t-t')\mathfrak{C}(\beta\cdot y)}-1\Big|\Big)D_0(x,y)
\end{aligned}$$

so that, as above $\sum_{\beta\in\mathfrak{B}}[w_{\delta+it}(\beta,\cdot)-w_{\delta+it'}(\beta,\cdot)]\preceq|t-t'|^\kappa L\left(\frac{1}{|t-t'|}\right)$. We achieve the proof of the Lemma combining this last inequality with (31). □

6.2. On the local expansion of the dominant eigenvalue λ_t . We explicit here the local expansion near 0 of the map $t\mapsto\lambda_t$:

PROPOSITION 6.6. *Under the hypotheses* $\mathbf{H_1}-\mathbf{H_3}$*, there exists* $C_\Gamma>0$ *such that*

$$\lambda_t=1-C_\Gamma e^{-i\pi\frac{\kappa}{2}}t^\kappa L(1/t)\ (1+o(t)). \tag{32}$$

Proof. Recall first that 1 is a simple eigenvalue of $\mathcal{L}_\delta$ with $\mathcal{L}_\delta h=h$ and $\sigma(h)=1$; since $t\mapsto\mathcal{L}_{\delta+it}$ is continuous on $\mathbb{R}$, for t closed to 0 there exists a function $h_t\in\mathbb{L}_\omega$ such that $\mathcal{L}_{\delta+it}h_t=\lambda_t h_t$, this function being unique if we impose the normalization condition $\sigma(h_t)=1$. The maps $t\mapsto\lambda_t$ and $t\mapsto h_t$ have the same type of continuity than $t\mapsto\mathcal{L}_{\delta+it}$ and one gets the identity

$$\lambda_t=\sigma\left(\mathcal{L}_t h_t\right)=\sigma\left(\mathcal{L}_t h\right)+\sigma\Big((\mathcal{L}_{\delta+it}-\mathcal{L}_\delta)(h_t-h)\Big)\Big).$$

By the previous subsection, the second term of this last expression is $\preceq\Big(t^\kappa L(1/t)\Big)^2$. It remains to precise the local behavior of the map $t\mapsto\sigma\left(\mathcal{L}_t h\right)$; one gets

$$\sigma\left(\mathcal{L}_t h\right)=1+\sum_{l=0}^{r}\sigma_l$$

with $\sigma_0:=\displaystyle\sum_{\beta\in\mathcal{W}_1}\int_{K_\beta}h(\beta\cdot x))e^{-\delta\mathfrak{C}(\beta\cdot x)}(e^{-it\mathfrak{C}(\beta\cdot x)}-1)\sigma(dx)$ and, for $1\leq l\leq r$

$$\sigma_l:=\sum_{\beta\in\widehat{\mathcal{P}}_l}\int_{K_\beta}h(\beta\cdot x))e^{-\delta\mathfrak{C}(\beta\cdot x)}(e^{-it\mathfrak{C}(\beta\cdot x)}-1)\sigma(dx).$$

By (30), the terms $\sigma_l, l\neq r$, are of the form $t^\kappa L(1/t)o(t)$. To control the term σ_r, one sets $\Delta(n,x):=\mathfrak{C}(p_r^n\cdot x)-d(\mathbf{x}_0,p_r^n\cdot\mathbf{x}_0)$ for any $x\in\overline{\partial X\setminus I_r}$ and $n\in\mathbb{Z}$. The following lemma readily implies that the quantity $\Delta(n,x)$ tends to $-(x|\xi_r)_{\mathbf{x_0}}$ as $|n|\to+\infty$.

LEMMA 6.7. *For any parabolic group* $\mathcal{P}:=<p>$ *with fixed point* ξ*, we have*

$$\mathcal{B}_x(p^{\pm n}\cdot\mathbf{x}_0,\mathbf{x}_0)=d_b(\mathbf{x}_0,p^n\cdot\mathbf{x}_0)-2(\xi|x)_{\mathbf{x}_0}+\epsilon_x(n)$$

with $\displaystyle\lim_{n\to+\infty}\epsilon_x(n)=0$*, the convergence being uniform on compact sets of* $\partial X\setminus\{\xi\}$*.*

Proof. Let $(\mathbf{x}_m)$ be a sequence of elements of X converging to x. We have

$$\begin{aligned}\mathcal{B}_x(p^{\pm n}\cdot\mathbf{x}_0,\mathbf{x}_0) &= \lim_m d(p^{\pm n}\cdot\mathbf{x}_0,\mathbf{x}_m)-d(\mathbf{x}_0,\mathbf{x}_m)\\ &= d(p^{\pm n}\cdot\mathbf{x}_0,\mathbf{x}_0)-\lim_m\Big(d(\mathbf{x}_0,\mathbf{x}_m)+d(p^{\pm n}\cdot\mathbf{x}_0,\mathbf{x}_0)-d(p^{\pm n}\cdot\mathbf{x}_0,\mathbf{x}_m)\Big)\end{aligned}$$

with

$$\lim_n\Big(\lim_m d(\mathbf{x}_0,\mathbf{x}_m)+d(p^{\pm n}\cdot\mathbf{x}_0,\mathbf{x}_0)-d(p^{\pm n}\cdot\mathbf{x}_0,\mathbf{x}_m)\Big)=2\lim_n(p^{\pm n}\cdot\mathbf{x}_0|x)_{\mathbf{x}_0}=2(\xi_h^-|x)_{\mathbf{x}_0}$$

and the conclusion follows as the Gromov product $(p^{\pm n}\cdot\mathbf{x}_0|x)_{\mathbf{x}_0}$ tends uniformly to $(\xi|x)_{\mathbf{x}_0}$ on compacts of $\mathbb{S}^1\setminus\{\xi\}$. □

We write

$$\begin{aligned}\sigma_r &= \sum_{|n|\geq 2}e^{-\delta d(\mathbf{x}_0,p_r^n\cdot\mathbf{x}_0)}\left(e^{-itd(\mathbf{x}_0,p_r^n\cdot\mathbf{x}_0)}-1\right)\int_{\partial X\setminus I_r}h(p_r^n\cdot x))e^{-\delta\Delta(n,x)}\sigma(dx)\\ &\quad+\sum_{|n|\geq 2}e^{-\delta d(\mathbf{x}_0,p_r^n\cdot\mathbf{x}_0)}\int_{\partial X\setminus I_r}h(p_r^n\cdot x))e^{-\delta\Delta(n,x)}\Big(e^{-it\mathfrak{C}(p_r^n\cdot x)}-e^{-itd(\mathbf{x}_0,p_r^n\cdot\mathbf{x}_0)}\Big)\sigma(dx)\\ &= \sigma_{r1}+\sigma_{r2}.\end{aligned}$$

One gets $\displaystyle\int_{\partial X\setminus I_r}h(p_r^n\cdot x))e^{-\delta\Delta(n,x)}\sigma(dx)>0$ for any $|n|\geq 2$; by (23), there exists $c>0$ such that

$$\sigma_{r1}=-ce^{-i\pi\frac{\kappa}{2}}\Gamma(1-\kappa)t^\kappa L\Big(\frac{1}{t}\Big)(1+o(t)).$$

On the other hand

$$|\sigma_{r2}| \leq |t| \sum_{|n|\geq 2} e^{-\delta d(\mathbf{x}_0, p_r^n\cdot \mathbf{x}_0)} \int_{\partial X\setminus I_r} h(p_r^n\cdot x))e^{-\delta\Delta(n,x)}|\mathfrak{C}(p_r^n\cdot x)-d(\mathbf{x}_0, p_r^n\cdot\mathbf{x}_0)|\sigma(dx) = O(t).$$

Equality (32) follows immediately. □

6.3. Renewal theory and proof of Theorem 6.1. For technical reasons (see for instance [**21**]) which will appear in the control of the term $\widetilde{W}_j^{(3)}(R,\psi)$, we need to symmetrize the quantities $W_j(R,\psi)$ and $W_j(s,R,\psi)$ setting

$$\widetilde{W}_j(R,\psi) := \sum_{k\geq 0} \sum_{y\in\partial X/T^k\cdot y=x_j} e^{-\delta S_k\mathfrak{C}(y)}\Big(\psi(S_k\mathfrak{C}(y)-R)+\psi(-S_k\mathfrak{C}(y)-R)\Big)$$

and

$$\widetilde{W}_j(s,R,\psi) := \sum_{k\geq 0} s^k \sum_{y\in\partial X/T^k\cdot y=x_j} e^{-\delta S_k\mathfrak{C}(y)}\Big(\psi(S_k\mathfrak{C}(y)-R)+\psi(-S_k\mathfrak{C}(y)-R)\Big).$$

Notice that, when ψ is a continuous function with compact support in $\mathbb{R}^+$, the terms $\psi(-S_k\mathfrak{C}(y)-R)$ of these sums vanish for R large enough, so that $\widetilde{W}_j(R,\psi) = W_j(R,\psi)$ and $\widetilde{W}_j(s,R,\psi) = W_j(s,R,\psi)$ in this case. By (21), for any $0<s<1$, one gets,

$$\begin{aligned}\widetilde{W}_j(s,R,\psi) &:= \sum_{k\geq 0} s^k \sum_{y\in\partial X/T^k\cdot y=x_j} e^{-\delta S_k\mathfrak{C}(y)}\Big(\psi(S_k\mathfrak{C}(y)-R)+\psi(-S_k\mathfrak{C}(y)-R)\Big)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\hat{\psi}(t)\Big((I-s\mathcal{L}_{\delta+it})^{-1}-(I-s\mathcal{L}_{\delta-it})^{-1}\Big)1(x_j)dt.\end{aligned}$$

Fix $\epsilon_0>0$ and let $\rho(t)$ be a symmetric C^∞-function on $\mathbb{R}$ which is equal to 1 on a $[-\epsilon_0,\epsilon_0]$ and which vanishes outside $[-2\epsilon_0, 2\epsilon_0]$; one thus decomposes $\widetilde{W}_j(s,R,\psi)$ as

$$\widetilde{W}_j(s,R,\psi) = \widetilde{W}_j^{(1)}(s,R,\psi)+\widetilde{W}_j^{(2)}(s,R,\psi)+\widetilde{W}_j^{(3)}(s,R,\psi)$$

with

$$\begin{aligned}\widetilde{W}_j^{(1)}(s,R,\psi) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\hat{\psi}(t)(1-\rho(t))\left(I-r\mathcal{L}_{\delta+it}\right)^{-1}1(x_j)dt\\ &\quad+\frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\hat{\psi}(t)(1-\rho(t))\left(I-r\mathcal{L}_{\delta-it}\right)^{-1}1(x_j)dt,\\ \widetilde{W}_j^{(2)}(s,R,\psi) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\hat{\psi}(t)\rho(t)\left(\left(I-r\mathcal{L}_{\delta+it}\right)^{-1}1(x_j)-\frac{\sigma(\partial X\setminus I_j)h(x_j)}{1-r\lambda_t}\right)dt\\ &\quad+\frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\hat{\psi}(t)\rho(t)\left(\left(I-r\mathcal{L}_{\delta-it}\right)^{-1}1(x_j)-\frac{\sigma(\partial X\setminus I_j)h(x_j)}{1-r\lambda_{-t}}\right)dt\\ \text{and } \widetilde{W}_j^{(3)}(s,R,\psi) &= \frac{\sigma(\partial X\setminus I_j)h(x_j)}{2\pi}\int_{\mathbb{R}} e^{itR}\hat{\psi}(t)\rho(t)\left(\frac{1}{1-r\lambda_t}-\frac{1}{1-r\lambda_{-t}}\right)dt.\end{aligned}$$

Using Proposition 6.6, letting $s\to 1$, one gets

$$\widetilde{W}_j(R,\psi) = \widetilde{W}_j^{(1)}(R,\psi)+\ \widetilde{W}_j^{(2)}(R,\psi)+\widetilde{W}_j^{(3)}(R,\psi)$$

with

$$\begin{aligned}
\widetilde{W}_j^{(1)}(R,\psi) &= \lim_{s\nearrow 1} \widetilde{W}_j^{(1)}(s,R,\psi)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\hat\psi(t)(1-\rho(t))\Big(\left(I-\mathcal{L}_{\delta+it}\right)^{-1}1(x_j)+\left(I-\mathcal{L}_{\delta-it}\right)^{-1}1(x_j)\Big)dt,\\
\widetilde{W}_j^{(2)}(R,\psi) &= \lim_{s\nearrow 1} W_j^{(2)}(s,R,\psi)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\hat\psi(t)\rho(t)\left(\left(I-\mathcal{L}_{\delta+it}\right)^{-1}1(x_j)-\frac{\sigma(\partial X\setminus I_j)h(x_j)}{1-\lambda_t}\right)dt\\
&\qquad +\frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\hat\psi(t)\rho(t)\left(\left(I-\mathcal{L}_{\delta-it}\right)^{-1}1(x_j)-\frac{\sigma(\partial X\setminus I_j)h(x_j)}{1-\lambda_{-t}}\right)dt\\
\text{and } \widetilde{W}_j^{(3)}(R,\psi) &= \lim_{s\nearrow 1} \widetilde{W}_j^{(3)}(s,R,\psi)\\
&= \frac{\sigma(\partial X\setminus I_j)h(x_j)}{2\pi}\int_{\mathbb{R}} e^{itR}\hat\psi(t)\rho(t)\mathrm{Re}\left(\frac{1}{1-\lambda_t}\right)dt.
\end{aligned}$$

The functions $t\mapsto\lambda_t$ has the same regularity as $t\mapsto L_{\delta+it}$; by lemma 6.4 one can thus check that $\psi_1 : t\mapsto \hat\psi(t)(1-\rho(t))\Big(\left(I-\mathcal{L}_{\delta+it}\right)^{-1}1(x_j)+\left(I-\mathcal{L}_{\delta-it}\right)^{-1}1(x_j)\Big)$ and

$$\begin{aligned}
\psi_2 : t &\mapsto \hat\psi(t)\rho(t)\Big(\left(I-\mathcal{L}_{\delta+it}\right)^{-1}1(x_j)-\frac{\sigma(\partial X\setminus I_j)h(x_j)}{1-\lambda_t}\\
&\qquad\qquad +\left(I-\mathcal{L}_{\delta-it}\right)^{-1}1(x_j)-\frac{\sigma(\partial X\setminus I_j)h(x_j)}{1-\lambda_{-t}}\Big)
\end{aligned}$$

satisfy the inequality $|\psi_k(s)-\psi_k(t)|\preceq |s-t|^\kappa L\left(\frac{1}{|s-t|}\right), k=1,2$. This yields some information on the speed of convergence to 0 of their Fourier transform : indeed, for any $\theta<\kappa$, there exists $C_\theta>0$ such that

$$\left|\widetilde{W}_j^{(1)}(R,\psi)\right|\le\frac{C_\theta}{R^\theta}\qquad\text{and}\qquad\left|\widetilde{W}_j^{(2)}(R,\psi)\right|\le\frac{C_\theta}{R^\theta}\tag{33}$$

(which readily implies $\lim_{R\to+\infty} R^{1-\kappa}L(R)\widetilde{W}_j^{(1)}(R,\psi)=\lim_{R\to+\infty} R^{1-\kappa}L(R)\widetilde{W}_j^{(2)}(R,\psi)=0$ as soon as $\theta\in]1-\kappa,\kappa[$, which is possible since $\kappa>1/2$).
On the other hand, by Section 5 in [**21**], one gets for $1/2<\kappa<1$

$$\lim_{R\to+\infty} R^{1-\kappa}L(R)\widetilde{W}_j^{(3)}(R,\psi)=C_j\hat\psi(0)=C_j\int_{\mathbb{R}}\psi(x)dx$$

with $C_j=\sigma(\partial X\setminus I_j)h(x_j)\frac{\sin\pi\kappa}{\pi}$; notice that the value $h(x_j)$ is uniquely determined by the normalization $\sigma(h)=1$ (see [**26**] Theorem 4 for a detailed argument). This achieves the proof of Theorem 6.1.□

Concluding remarks.
(1) As it is clear from the proof, we do not really need that the metric g is hyperbolic outside the cusps $\bar{\mathcal{C}}_i$; hyperbolicity is only needed to describe the initial fundamental polygon $\mathcal{D}$ with a fundamental system of horospheres. As far as the metric g on $\bar X$ has negative bounded by a negative constant from above (hence, the conformal structure of ∂X is well defined and the related properties as in 4.2, 4.4, 4.6 are satisfied) and satisfies the conditions $\mathbf{H_1}-\mathbf{H_3}$, all the arguments in the proof of Theorem continue to hold.
(2) The same estimates for the orbital function can be deduced for surfaces with $r+1$ punctures and genus $g>0$, which have *essentially free* fundamental group Γ, according to the definition in [**35**]. This can be achieved by using a different coding as given in [**35**] (which cannot be used in our case, where adjacent sides are paired by a parabolic element). We limited ourselves to genus 0 surfaces as our coding is particularly simple and explicit in this case.

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