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Some negatively curved manifolds with cusps, mixing and counting

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Abstract. Let X be a Hadamard manifold whose sectional curvature K satisfies $-b^2 \leqq K \leqq -1$. We consider a family of free isometry groups Γ acting properly discontinuously on X and containing parabolic transformations of divergence type. We show that such groups are of divergent type, we describe the dynamic properties of the map T induced by the action of Γ on the boundary of X and we explore the spectrum of the transfer operator associated with T. As applications, we establish a mixing property for the geodesic flow on the unit tangent bundle of X/Γ and we describe the behaviour as a goes to $+\infty$ of the number of primitive closed geodesics on X/Γ whose length is not larger than a.

Introduction

A *Hadamard manifold* is a complete simply connected Riemannian manifold of non positive curvature K. Let X be such a manifold, assume that $-b^2 \leqq K \leqq -1$ and denote by ∂X its visual boundary relatively to a reference point 0. Fix two integers N_1, N_2 such that $N_1 + N_2 \geqq 2$ and $N_2 \geqq 1$ and consider N_1 hyperbolic isometries $\alpha_1, \ldots, \alpha_{N_1}$ and N_2 parabolic ones $\alpha_{N_1+1}, \ldots, \alpha_{N_1+N_2}$ satisfying the following conditions:

(1) *For $1 \leqq i \leqq N_1$ there exist in ∂X a compact neighbourhood C_{α_i} of the attracting point x_{α_i} of α_i and a compact neighbourhood $C_{\alpha_i^{-1}}$ of the reppeling point $x_{\alpha_i^{-1}}$ such that*

$$\alpha_i(\partial X - C_{\alpha_i^{-1}}) \subset C_{\alpha_i}.$$

(2) *For $N_1 + 1 \leqq i \leqq N_1 + N_2$ there exists in ∂X a compact neighbourhood C_{α_i} of the unique fixed point x_{α_i} of α_i such that*

$$\forall n \in \mathbb{Z}^* \quad \alpha_i^n(\partial X - C_{\alpha_i}) \subset C_{\alpha_i}.$$

(3) *The $2N_1 + N_2$ neighbourhoods introduced in* (1) *and* (2) *are pairwise disjoint.*

(4) *The elementary groups $\langle \alpha_i \rangle$ for $N_1 + 1 \leqq i \leqq N_1 + N_2$ are of divergence type.*

Such families of isometries can be obtained by taking some powers of a finite number of parabolic or hyperbolic transformations of divergence type which have no fixed points in common. Note that if $N_1 = 0$ we only consider $N_2 \geqq 2$ parabolic transformations satisfying conditions (2) and (3).

Transformations $\alpha_1, \ldots, \alpha_{N_1+N_2}$ generate a free group Γ which acts properly discontinuously and freely on X. If $N_2 = 0$ the group Γ is a *Schottky* group; it acts on the convex hull of its limit set with compact fundamental domain, in other words Γ is *convex cocompact*. The geometry of convex cocompact groups is well known ([7], [19], [33]); many results are proved using the thermodynamic formalism which may be applied in this case.

Throughout the present paper we assume that $N_2 \geqq 1$ and we will say that Γ is an *extended Schottky* group. Note that any parabolic transformation of Γ is conjugated to some power of a parabolic generator; this is the simplest example of a non convex-cocompact group.

In the case where $X = \mathbb{H}^n_{\mathbb{R}}$, there is a well known ergodic theory for geometrically finite discret groups G even if it contains parabolic transformations: for example the Patterson-Sullivan measure associated with G has no atomic part, G is of divergence type [31] and the geodesic flow on the unit tangent bundle of X/Γ is topologically mixing [28].

In the non constant curvature case, there are not many results about groups with parabolic transformations and several problems are still open: Are such groups of divergence type? Does there exist an atomic part in the Patterson-Sullivan measure? Is the geodesic flow mixing relatively to the geometrical Patterson-Sullivan measure?

Let us now state the main results of the present paper. Denote by d the Riemannian distance on X and δ_Γ the exponent of convergence of the Poincaré series associated with Γ. Since the sectional curvature of X is lower bounded, δ_Γ is finite. Let g be a non elliptic isometry, denote by δ_g the exponent of convergence of the series $\sum_{n\in\mathbb{Z}} e^{-sd(0,g^n 0)}$. If g is hyperbolic then $\delta_g = 0$; if g is parabolic then $\delta_g \geqq \frac{1}{2}$ (cf. § III) We prove the

Theorem III.1. *Let G be a non elementary group of isometries of X. For any $g \in G$ such that $\sum_{n\in\mathbb{Z}} e^{-\delta_g d(0,g^n 0)} = +\infty$, one has $\delta_G > \delta_g$.*

When g is hyperbolic one just obtains the well known result $\delta_G > 0$. If G contains parabolic transformations then $\delta_G > 1/2$.

The following results are stated for extended Schootky groups Γ; using the coding of the limit set of these groups we prove the

Theorem IV.2. *The Patterson-Sullivan measure σ associated with Γ has no atomic part.*

As a direct consequence we obtain

Corollary IV.3. *The group Γ is of divergence type, that is* $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d(0,\gamma 0)} = +\infty$.

Since Γ contains parabolic transformations, we code the points of its radial limit set Λ^0 with an infinite alphabet; by geometrical arguments, we show in § V that the boundary map T on Λ^0 induced by the classical shift on the associated symbolic space is expanding and we construct a T-invariant probability measure ν on Λ^0.

Denote by Λ the limit set of Γ and by $G\Lambda$ the set of pairs (ξ, x) where ξ is a geodesic on X with endpoints in Λ and $x \in \xi$. The Patterson-Sullivan measure induces a natural measure $\overline{\mu \otimes l}$ on the non wandering set $G\Lambda/\Gamma$ of the geodesic flow $(\bar{g}_t)_{t\in\mathbb{R}}$ on the unit tangent bundle of X which is $(\bar{g}_t)_{t\in\mathbb{R}}$ invariant. We show the

Theorem VI.2. *The geodesic flow $(\bar{g}_t)_{t\in\mathbb{R}}$ on $G\Lambda/\Gamma$ is mixing relatively to $\overline{\mu \otimes l}$.*

The proof of this result is based on a renewal theorem for transient Markov walk on $\Lambda \times \mathbb{R}$ [14] and requires a precise investigation of the spectrum of the adjoint operator P and its Fourier transforms P_λ associated with T and ν (cf. § VIII); the fact that Γ contains parabolic isometries is essential to describe the top of the spectrum of P_λ, $\lambda \in \mathbb{R}$. Note that, as far as we know, theorem VI.2 is not proved (or not yet published) in the case where the sectional curvature of X is non constant and Γ is convex cocompact, even if Γ is a Schottky group.

For any $a > 0$ denote by $\pi(a)$ the number of primitive closed geodesics on X/Γ with length not larger than a; since Γ is not purely hyperbolic the set of closed geodesics on X/Γ is not relatively compact. In § VII we prove the following

Theorem VII.1. *The function $a \to \pi(a)$ is equivalent to $e^{a\delta_\Gamma}/a\delta_\Gamma$ as a goes to $+\infty$.*

To prove this theorem we use a probabilistic method introduced by S. Lalley [21] and already developped in further directions ([2], [6], and [22]). First we code closed geodesics on X/Γ and establish a connection between $\pi(a)$ and the harmonic potential of a certain Markov walk on $\mathbb{R}$. Theorem VII.1 thus appears as a direct consequence of a harmonic renewal theorem for a transient Markov walk on $\mathbb{R}$.

When $M = X/\Gamma$ is compact, theorem VII.1 is due to Selberg [29] ($K = -1$), Margulis [23] (K variable) and has been extended to periodic orbits of Axiom A flows in [25]. When M is not compact and $K = -1$ this theorem is well known if M is a surface ([13], [17], [32]); when $\dim M \geqq 3$ a similar result holds under the following hypotheses: the volume of M is finite [32], or $\pi_1(M)$ is convex cocompact ([21], [25]) or $\pi_1(M)$ is a *Ping-Pong group* [10]. The case where M is not compact and the curvature K is not constant is quite open; in [6] we solve it for small perturbations of the Poincaré metric on the modular surface.

Remark. All the results of the present paper are valid and the proofs are rigorously the same if one replaces X by a CAT(-1)-space whose boundary has a finite visual Hausdorff dimension [27] and admits hyperbolic and parabolic isometries; unfortunately, we have no interesting example of such spaces except pinched Hadamard manifolds.

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I. Geometry on X

Denote by d the Riemannian distance on X. Since the sectional curvature is not larger than -1 the metric space (X, d) is a CAT(-1)-space [4].

Throughout this paper we fix a reference point $0 \in X$. Consider two geodesic rays $t \to r(t)$ and $t \to s(t)$ based at 0; one says that r and s are equivalent if the Hausdorff distance between r and s is bounded. Denote by ∂X the quotient of the set of geodesic rays based at 0 by this equivalence relation; equipped with the quotient topology induced by uniform convergence on compacts, $X \cup \partial X$ is compact. Note that every pair of distinct points in $X \cup \partial X$ determines a unique geodesic on X [4].

A metric $d_{\partial X}$ on ∂X is called a *visual t metric* (with $t > 1$) if there exists $C \geqq 1$ such that for every $x, y \in \partial X$ one has $\frac{1}{C} t^{-d(0,(xy))} \leqq d_{\partial X}(x, y) \leqq C t^{-d(0,(xy))}$ where $d\big(0, (xy)\big)$ denotes the distance from 0 to the geodesic (xy) joining x and y. Such a metric does exist on the boundary of any Gromov hyperbolic space [7], hence in particular on the boundary of a CAT(-1)-space. Let $x \in \partial X$ and $t \to r(t)$ be a geodesic ray joigning 0 and x; for every $z_1, z_2 \in X$ the limit as t goes to $+\infty$ of the difference $d\big(z_1, r(t)\big) - d\big(z_2, r(t)\big)$ exists and is denoted by $B_x(z_1, z_2)$. Geometrically, $B_x(z_1, z_2)$ represents the algebraic horospherical distance between z_1 and z_2 relatively to x; moreover, if $x_1, x_2 \in \partial X$ and z belongs to the geodesic with extremities x_1 and x_2 then $(x_1|x_2) = \frac{B_{x_1}(0, z) + B_{x_2}(0, z)}{2}$ does not depend on the choice of z. One has the

Theorem I.1 ([4]). *The mapping $D : \partial X \times \partial X \to \mathbb{R}^+$ defined by $D(x_1, x_2) = e^{-(x_1|x_2)}$ if $x_1 \neq x_2$ and $D(x_1, x_1) = 0$ is a visual e-metric on ∂X.*

For every isometry γ on X and every point $x \in \partial X$ set $|\gamma'(x)| = e^{B_x(0, \gamma^{-1} 0)}$. Using the equalities $B_{\gamma(x)}(\gamma z_1, \gamma z_2) = B_x(z_1, z_2)$ and $B_x(z_1, z_3) = B_x(z_1, z_2) + B_x(z_2, z_3)$ one obtains the

Mean values relation. $\forall x, y \in \partial X \quad D(\gamma x, \gamma y) = \sqrt{|\gamma'(x)||\gamma'(y)|}\, D(x, y).$

Isometries on X are classified according to their fix points. An isometry is *elliptic* if it has a fixed point inside X; in the present paper we only consider non elliptic isometries γ. Thus

– either γ fixes a unique point $x_\gamma \in \partial X$; in this case γ is said to be *parabolic* and $|\gamma'(x_\gamma)| = 1$;

– or γ fixes exactly two distinct points x_γ and $x_{\gamma^{-1}}$; in this case, γ is said to be *hyperbolic*, the point x_γ satisfies the inequality $|\gamma'(x_\gamma)| < 1$ and is called the *attracting point* of γ, the point $x_{\gamma^{-1}}$ satisfies the equality $|\gamma'(x_{\gamma^{-1}})| = \frac{1}{|\gamma'(x_\gamma)|} > 1$ and is thus called the *repelling point* of γ. Near $x_{\gamma^{-1}}$ the hyperbolic isometry γ looks like a homothety expansion with *expansion rate* $|\gamma'(x_{\gamma^{-1}})|$.

Definition I.2. Let γ be a hyperbolic isometry acting on X; the expansion rate of γ is the real number $\Phi(\gamma) = |\gamma'(x_{\gamma^{-1}})|$.

The two following lemmas describe the dynamic on ∂X of non elliptic isometries.

Lemma I.3. *Let γ be a hyperbolic isometry; for every compact set $E \subset \partial X - \{x_\gamma, x_{\gamma^{-1}}\}$ there exists $A \geqq 1$ depending on γ and E such that*

(1) $\forall x \in E,\ \forall n \in \mathbb{Z}^* \quad \frac{\Phi(\gamma)^{-|n|}}{A} \leqq |(\gamma^n)'(x)| \leqq A\Phi(\gamma)^{-|n|}$,

(2) $\forall x, y \in E,\ \forall n \in \mathbb{Z}^* \quad \big||(\gamma^n)'(x)| - |(\gamma^n)'(y)|\big| \leqq A\Phi(\gamma)^{-|n|} D(x, y)$.

Proof. Assume $n \geqq 1$; if $n \leqq -1$ it suffices to replace $x_{\gamma^{-1}}$ by x_γ in the proof.

(1) One has $D(\gamma^n x, x_{\gamma^{-1}}) = \sqrt{|(\gamma^n)'(x)|\,|\gamma'(x_{\gamma^{-1}})|^n}\, D(x, x_{\gamma^{-1}})$. Suppose first that for every $k \geqq 1$ there exist $x_k \in E$ and $p(k) \in \mathbb{N}^*$ such that $D(\gamma^{p(k)} x_k, x_{\gamma^{-1}}) \leqq 1/k$; therefore $\lim_{k \to +\infty} \gamma^{p(k)}(x_k) = x_{\gamma^{-1}}$. On the other hand

$$\frac{D(\gamma^{p(k)} x_k, x_{\gamma^{-1}})}{D(\gamma^{p(k)} x_k, x_\gamma)} = \Phi(\gamma)^{p(k)} \frac{D(x_k, x_{\gamma^{-1}})}{D(x_k, x_\gamma)} \geqq \frac{D(E, x_{\gamma^{-1}})}{\|D\|_\infty}$$

which implies $\lim_{k \to +\infty} \gamma^{p(k)}(x_k) = x_\gamma$; this contradicts the fact that $x_\gamma \neq x_{\gamma^{-1}}$. Consequently, there exists $B > 0$ such that for every $x \in E$ and $n \geqq 1$ one has $D(\gamma^n x, x_{\gamma^{-1}}) \geqq B$. By the mean values relation it follows

$$\left(\frac{B}{\|D\|_\infty}\right)^2 \Phi(\gamma)^{-n} \leqq |(\gamma^n)'(x)| \leqq \left(\frac{\|D\|_\infty}{D(E, x_{\gamma^{-1}})}\right)^2 \Phi(\gamma)^{-n}.$$

(2) Let x and y be in ∂X. Set $\lambda(x, y) = \||(\gamma^n)'(x)| - |(\gamma^n)'(y)|\|$; one has

$$\begin{aligned}\lambda(x, y) &= \Phi(\gamma)^{-n} \left| \frac{D^2(\gamma^n x, x_{\gamma^{-1}})}{D^2(x, x_{\gamma^{-1}})} - \frac{D^2(\gamma^n y, x_{\gamma^{-1}})}{D^2(y, x_{\gamma^{-1}})} \right| \\ &\leqq \frac{\Phi(\gamma)^{-n}}{D^2(x, x_{\gamma^{-1}})} |D^2(\gamma^n x, x_{\gamma^{-1}}) - D^2(\gamma^n y, x_{\gamma^{-1}})| \\ &\quad + \Phi(\gamma)^{-n} D^2(\gamma^n y, x_{\gamma^{-1}}) \left| \frac{1}{D^2(x, x_{\gamma^{-1}})} - \frac{1}{D^2(y, x_{\gamma^{-1}})} \right|.\end{aligned}$$

If x and y belong to E one obtains

$$\begin{aligned}\lambda(x,y) &\leqq \Phi(\gamma)^{-n}\frac{2\|D\|_\infty}{D^2(E,x_{\gamma^{-1}})}D(\gamma^n x,\gamma^n y)\\&\quad+\Phi(\gamma)^{-n}\frac{2\|D\|_\infty^3}{D^4(E,x_{\gamma^{-1}})}D(x,y)\\&\leqq \Phi(\gamma)^{-n}\frac{2\|D\|_\infty}{D^2(E,x_{\gamma^{-1}})}\sqrt{|(\gamma^n)'(x)|\,|(\gamma^n)'(y)|}\,D(x,y)\\&\quad+\Phi(\gamma)^{-n}\frac{2\|D\|_\infty^3}{D^4(E,x_{\gamma^{-1}})}D(x,y).\end{aligned}$$

Using inequality (1) one thus obtains the existence of a constant A such that

$$0\leqq\lambda(x,y)\leqq A\Phi(\gamma)^{-n}D(x,y). \quad \square$$

The dynamic of parabolic isometries is a little different. One has the

Lemma I.4. *Let γ be a parabolic isometry; for every compact set $E\subset\partial X-\{x_\gamma\}$ and every $y_0\in E$ there exists $A\geqq 1$ depending on γ and E*

(1) $\forall x\in E, \forall n\in\mathbb{Z}^*\quad \dfrac{|(\gamma^n)'(y_0)|}{A}\leqq|(\gamma^n)'(x)|\leqq A|(\gamma^n)'(y_0)|,$

(2) $\forall x,y\in E, \forall n\in\mathbb{Z}^*\quad \big||(\gamma^n)'(x)|-|(\gamma^n)'(y)|\big|\leqq A|(\gamma^n)'(y_0)|D(x,y).$

Furthermore one has $\lim\limits_{n\to\pm\infty}|(\gamma^n)'(y_0)|^{1/|n|}=1.$

Proof. (1) Since $|\gamma'(x_\gamma)|=1$ one has $D(\gamma^n x,x_\gamma)=\sqrt{|(\gamma^n)'(x)|}\,D(x,x_\gamma)$ for any $x\in\partial X$; so $\dfrac{1}{\|D\|_\infty^2}D^2(\gamma^n x,x_\gamma)\leqq|(\gamma^n)'(x)|\leqq\dfrac{1}{D^2(x_\gamma,E)}D^2(\gamma^n x,x_\gamma)$. For every $x,y\in\partial X-\{x_\gamma\}$ set $\Delta(x,y)=\dfrac{D(x,y)}{D(x,x_\gamma)D(y,x_\gamma)}$; using the fact that $\Delta(\gamma x,\gamma y)=\Delta(x,y)$ one obtains $\left|\dfrac{1}{D(\gamma^n x,x_\gamma)}-\dfrac{1}{D(\gamma^n y,x_\gamma)}\right|\leqq\Delta(x,y)$ for any $n\in\mathbb{Z}^*$. Now, fix $y_0\in E$; we have $\|\Delta\|_\infty=\sup\limits_{x\in E}\Delta(x,y_0)<+\infty$ and so

$$\forall x\in E\quad \frac{D(\gamma^n y_0,x_\gamma)}{1+\|\Delta\|_\infty D(\gamma^n y_0,x_\gamma)}\leqq D(\gamma^n x,x_\gamma)\leqq\frac{D(\gamma^n y_0,x_\gamma)}{1-\|\Delta\|_\infty D(\gamma^n y_0,x_\gamma)}.$$

Since $\lim\limits_{n\to\pm\infty}D(\gamma^n y_0,x_\gamma)=0$ there exists $C\geqq 1$ such that

$$\frac{1}{C}D(\gamma^n y_0,x_\gamma)\leqq D(\gamma^n x,x_\gamma)\leqq CD(\gamma^n y_0,x_\gamma)$$

and so

$$\frac{D^2(y_0, x_\gamma)}{\|D\|_\infty^2 C^2} |(\gamma^n)'(y_0)| \leqq |(\gamma^n)'(y)| \leqq \frac{C^2 D^2(y_0, x_\gamma)}{D^2(x_\gamma, E)} |(\gamma^n)'(y_0)|.$$

(2) Let x and y in ∂X and set $\lambda(x, y) = \big||(\gamma^n)'(x)| - |(\gamma^n)'(y)|\big|$; one has

$$\begin{aligned}\lambda(x, y) &= \left| \frac{D^2(\gamma^n x, x_\gamma)}{D^2(x, x_\gamma)} - \frac{D^2(\gamma^n y, x_\gamma)}{D^2(y, x_\gamma)} \right| \\ &\leqq \frac{1}{D^2(x, x_\gamma)} |D^2(\gamma^n x, x_\gamma) - D^2(\gamma^n y, x_\gamma)| \\ &\quad + D^2(\gamma^n y, x_\gamma) \left| \frac{1}{D^2(x, x_\gamma)} - \frac{1}{D^2 y, x_\gamma)} \right|.\end{aligned}$$

If x and y belong to E one obtains

$$\begin{aligned}\lambda(x, y) &\leqq \frac{2\|D\|_\infty}{D^2(E, x_\gamma)} D(\gamma^n x, \gamma^n y) \\ &\quad + \frac{2\|D\|_\infty}{D^4(E, x_\gamma)} D^2(\gamma^n y, x_\gamma) D(x, y) \\ &\leqq \frac{2\|D\|_\infty}{D^2(E, x_\gamma)} \sqrt{|(\gamma^n)'(x)||(\gamma^n)'(y)|} D(x, y) \\ &\quad + \frac{2\|D\|_\infty^3}{D^4(E, x_\gamma)} |(\gamma^n)'(y)| D(x, y).\end{aligned}$$

Using inequality (1) one thus obtains the existence of a constant $A \geqq 1$ such that $0 \leqq \lambda(x, y) \leqq A |(\gamma^n)'(y_0)| D(x, y)$.

Finally, since $\lim_{|n| \to +\infty} \frac{|(\gamma^{n+1})'(y_0)|}{|(\gamma^n)'(y_0)|} = \lim_{|n| \to +\infty} |\gamma'(\gamma^n y_0)| = 1$ we have

$$\lim_{|n| \to +\infty} |(\gamma^n)'(y_0)|^{1/|n|} = 1. \quad \square$$

The following result will be useful in the sequel; its proof is similar to the one of proposition 8, chap. 8 in [12].

Lemma I.5. *Let $(\gamma_n)_{n \geqq 1}$ be a sequence of pairwise distinct isometries of X such that $(\gamma_n(z_0))_{n \geqq 1}$ converges to a point $x \in \partial X$ for some $z_0 \in X$. Then for every z in X one has $\lim_{n \to +\infty} \gamma_n(z) = x$.*

The action of the sequence $(\gamma_n)_{n \geqq 1}$ on ∂X is a little different; for example if γ is a hyperbolic isometry, $x_{\gamma^{-1}}$ is fixed by γ and $\lim_{n \to +\infty} \gamma^n(x) = x_\gamma$ for all $x \neq x_{\gamma^{-1}}$. More generally one has

Corollary I.6. *Let $(\gamma_n)_{n \geqq 1}$ be a sequence of pairwise distinct isometries of X such that $(\gamma_n(z_0))_{n \geqq 1}$ converges to a point $x \in \partial X$ for some $z_0 \in X$. Then, for any couple (y, y') of distinct points in ∂X one has* $\lim_{n \to +\infty} \inf(D(\gamma_n y, x), D(\gamma_n y', x)) = 0$.

II. Extended Schottky groups

Fix two integers N_1 and N_2 such that $N_1 + N_2 \geqq 2$ and $N_2 \geqq 1$ and consider N_1 hyperbolic isometries $\alpha_1, \ldots, \alpha_{N_1}$ and N_2 parabolic ones $\alpha_{N_1+1}, \ldots, \alpha_{N_1+N_2}$ satisfying the conditions (1), (2), (3) and (4) given in the introduction. The group Γ generated by $\alpha_1, \ldots, \alpha_{N_1+N_2}$ is called an *extended Schottky group*.

Notations. Denote by $\mathscr{A} = \{\alpha_1, \ldots, \alpha_{N_1+N_2}\}$ and $\mathscr{A}^{\pm} = \{\alpha_1, \alpha_1^{-1}, \ldots, \alpha_{N_1+N_2}, \alpha_{N_1+N_2}^{-1}\}$. For every $1 \leqq i \leqq N_1$ set $C_{\alpha_i^{\pm}} = C_{\alpha_i} \cup C_{\alpha_i^{-1}}$ and for every $N_1 + 1 \leqq i \leqq N_1 + N_2$ set $C_{\alpha_i^{\pm}} = C_{\alpha_i}$.

Using conditions (1), (2) and (3) one shows by induction over n the

Property II.1 (Ping-Pong property). *Let $a_1, \ldots, a_n \in \mathscr{A}^{\pm}$ such that $a_{i+1} \neq a_i^{-1}$ for $1 \leqq i < n$. Then $a_1 \cdots a_n(\partial X - C_{a_n^{-1}}) \subset C_{a_1}$.*

Consequently $\mathscr{A}$ is a free system of generators of Γ. Furthermore one has

Corollary II.2. *The group Γ acts properly discontinuously on X.*

Proof. Assume that Γ does not act properly discontinuously on X. So there exists a sequence $(\gamma_n)_{n \geqq 1}$ of distinct elements of Γ such that $(\gamma_n(0))_{n \geqq 1}$ remains bounded. Fix two distinct points x and y in $\partial X - \bigcup_{a \in \mathscr{A}} C_{a^{\pm}}$ and choose z and z' on the geodesic (xy); since $(\gamma_n(z))_{n \geqq 1}$ and $(\gamma_n(z'))_{n \geqq 1}$ remain bounded, there exists a subsequence $(\gamma_{n_k})_{k \geqq 1}$ of $(\gamma_n)_{n \geqq 1}$ such that $(\gamma_{n_k}(z))_{k \geqq 1}$ and $(\gamma_{n_k}(z'))_{k \geqq 1}$ converge in X. Set $g_k = \gamma_{n_{k+1}}^{-1} \gamma_{n_k}$; one has $\lim_{k \to +\infty} g_k(z) = z$ and $\lim_{k \to +\infty} g_k(z') = z'$ so that $\lim_{k \to +\infty} g_k(x) = x$ and $\lim_{k \to +\infty} g_k(y) = y$. By the Ping-Pong property it follows that $g_k = \mathrm{Id}$ for k large enough which contradicts the hypothesis. □

Recall that $\Phi(\gamma) = |\gamma'(x_{\gamma^{-1}})|$ for every isometry γ.

Corollary II.3. *Let $(\gamma_n)_{n \geqq 1}$ be a sequence of distinct hyperbolic isometries of Γ such that γ_n and γ_m are not conjugated for every $n \neq m$. Then* $\lim_{n \to +\infty} \Phi(\gamma_n) = +\infty$.

Proof. Suppose that there exist $B > 0$ and a subsequence (denoted also $(\gamma_n)_{n \geqq 1}$) such that $\Phi(\gamma_n) \leqq B$ for every $n \geqq 1$. Since $\mathscr{A}$ is finite and $\Phi(g\gamma_n g^{-1}) = \Phi(\gamma_n)$ for any isometry g, one can suppose that for n large enough, either $\gamma_n = \alpha_i^{k_n}$ for some $1 \leqq i \leqq N_1$, or $\gamma_n = a_{n_1} \cdots a_{nk_n}$ with $a_{n_i} \in \mathscr{A}^{\pm}$, $a_{n(i+1)} \neq a_{ni}^{-1}$ for $1 \leqq i < k_n$ and $a_{n_1} = \alpha$, $a_{nk_n} = \beta$, $\alpha \neq \beta$.

In the first case one has $\Phi(\gamma_n) = \Phi(\alpha_i)^{|k_n|}$ with $\Phi(\alpha_i) > 1$ and corollary II.3 follows. In the second case $x_{\gamma_n^{-1}} \in C_{\beta^{\pm}}$ and $\gamma_n C_{\alpha^{\pm}} \subset C_{\alpha^{\pm}}$. For every $x \in C_{\alpha^{\pm}}$ one thus has

$|(\gamma_n)'(x)| \geqq \dfrac{D^2(C_{\alpha^\pm}, C_{\beta^\pm})}{B\|D\|_\infty^2}$ and therefore $\inf_{n\geqq 1} D(\gamma_n x, \gamma_n x') > 0$ for every pair (x, x') of distinct points in $C_{\alpha^\pm}$, which contradicts corollary I.6. □

The following lemma is important in order to code some limit points of Γ. For every subset $E \subset \partial X$ denote by $\operatorname{diam} E = \sup_{\substack{x, y \in E \\ x \neq y}} D(x, y)$. One has

Lemma II.4. *Let $(a_i)_{i\geqq 1} \in (\mathscr{A}^\pm)^{\mathbb{N}^*}$ such that $a_{i+1} \neq a_i^{-1}$ for every $i \geqq 1$. Then*

$$\lim_{n\to+\infty} \operatorname{diam} a_1 \cdots a_n \Big(\bigcup_{a \in \mathscr{A} - \{a_n^{-1}\}} C_a \Big) = 0 .$$

Proof. Since the sequence $\Big(a_1 \cdots a_n \big(\bigcup_{a \in \mathscr{A} - \{a_n^{-1}\}} C_a \big)\Big)_{n\geqq 1}$ is decreasing, it suffices to show the lemma for some subsequence. For every $n \geqq 1$ set $\gamma_n = a_1 \cdots a_n$ and consider a subsequence $(\gamma_{n_k})_{k\geqq 1}$ such that $a_{n_k} = \alpha \neq a_1^{\pm 1}$ (if such a subsequence does not exist replace the initial sequence $a_1, a_2, \dots$ by $a_1', a_2, \dots$ with $a_1' \neq a_1^{\pm 1}$ and $a_1' \neq a_2^{-1}$); transformations γ_{n_k} are thus hyperbolic and are not mutually conjugated. One has

$$\begin{aligned}
\operatorname{diam} \gamma_{n_k} \Big(\bigcup_{a \in \mathscr{A} - \{a^{-1}\}} C_a \Big) &\leqq \sup_{x \in \bigcup_{a \neq \alpha^{-1}} C_a} |\gamma_{n_k}'(x)| \quad \|D\|_\infty \\
&\leqq \sup_{x \in \bigcup_{a \neq \alpha^{-1}} C_a} \frac{D^2(\gamma_{n_k} x, x_{\gamma_{n_k}^{-1}})}{\Phi(\gamma_{n_k}) D^2(x, x_{\gamma_{n_k}^{-1}})} \quad \|D\|_\infty \\
&\leqq \frac{\|D\|_\infty^3}{\Phi(\gamma_{n_k}) D^2\Big(\bigcup_{a \in \mathscr{A} - \{\alpha^{-1}\}} C_a, C_{\alpha^{-1}} \Big)} .
\end{aligned}$$

By corollary II.3 one has $\lim_{n\to+\infty} \Phi(\gamma_{n_k}) = +\infty$ which finishes the proof. □

Denote by Λ the limit set of Γ; by definition $\Lambda = \overline{\Gamma 0} \cap \partial X$ and it is the least Γ-invariant closed subset of ∂X. Let us introduce the

Notations. For every $a \in \mathscr{A}^\pm$ set $\Lambda_a = \Lambda \cap C_a$ and $\Lambda_{a^\pm} = \Lambda \cap C_{a^\pm}$.

Let Λ^0 be the limit set Λ minus the Γ-orbit of the fixed points of $\alpha_1, \dots, \alpha_{N_1+N_2}$ and set $\Lambda_a^0 = \Lambda^0 \cap C_a$ and $\Lambda_{a^\pm}^0 = \Lambda^0 \cap C_{a^\pm}$.

Fix $x_0 \in \partial X - \bigcup_{a \in \mathscr{A}} C_{a^\pm}$. Let $x \in \Lambda^0$; since x is a limit point there exists a sequence $(\gamma_n)_{n\geqq 1}$ of Γ such that $\lim_{n\to+\infty} \gamma_n(x_0) = x$. Since $\mathscr{A}$ is finite and $x \notin \Gamma x_\alpha$ for any $\alpha \in \mathscr{A}^\pm$ there exists a subsequence $(\gamma_{n_k})_{k\geqq 1}$ of $(\gamma_n)_{n\geqq 1}$ such that $\gamma_{n_k} = a_1^{n_1} \cdots a_{l(k)}^{n_{l(k)}}$ with $a_i \in \mathscr{A}$, $n_i \in \mathbb{Z}^*$ and $a_{i+1} \neq a_i$. The unicity of the sequence $(a_i^{n_i})_{i\geqq 1}$ is a direct consequence of the Ping-Pong property. We therefore have

Property II.5 (coding property). *Fix $x_0 \in \partial X - \bigcup_{a \in \mathscr{A}} C_{a^\pm}$. For every $x \in \Lambda^0$ there exists an unique sequence $\omega(x) = (a_i^{n_i})_{i\geqq 1}$ with $a_i \in \mathscr{A}$, $n_i \in \mathbb{Z}^*$ and $a_{i+1} \neq a_i$ such that $\lim_{k\to+\infty} a_1^{n_1} \cdots a_k^{n_k} x_0 = x$.*

III. The critical gap property

Let g be a non-elliptic isometry on X and denote by δ_g the exponent of convergence of the Poincaré series $\sum_{n\in\mathbb{Z}} e^{-sd(0,g^n0)}$. If g is hyperbolic, replace 0 by a point z which belongs to the axis of g; one obtains $\sum_{n\in\mathbb{Z}} e^{-sd(z,g^nz)} = \sum_{n\in\mathbb{Z}} e^{-snB_{x_g}(0,g0)}$. Since $B_{x_g}(0,g0)>0$, one has $\delta_g=0$ and the group generated by g is of divergence type.

If g is parabolic, the estimate of δ_g is much more complicated. For example, a direct computation shows that $\delta_g=1/2$ when X is the real hyperbolic half space and $\delta_g\in\{1/2,1\}$ when X is the complex hyperbolic half space (*we thank here M. Bourdon who has pointed out this fact to us*). In the general case where X is a Hadamard manifold with sectional curvature $-b^2\leqq K\leqq -1$, denote by $\mathscr{H}$ the horosphere through 0 and with center x_g and by h the distance in $\mathscr{H}$ with respect to the induced metric; for any $p,q\in\mathscr{H}$ one has $2\sinh\frac{d(p,q)}{2}\leqq h(p,q)\leqq\frac{2}{b}\sinh\frac{b}{2}d(p,q)$ ([16], Thm. 4.6). Since $g^n(0)$ belongs to $\mathscr{H}$ for any $n\in\mathbb{Z}$ one has

$$\begin{aligned} d(0,g^n0) &\leqq 2\,\mathrm{Log}\big(h(0,g^n0)+1\big)\\ &\leqq 2\,\mathrm{Log}\big(h(0,g0)+\cdots+h(g^{n-1}0,g^n0)+1\big)\\ &\leqq 2\,\mathrm{Log}\big((|n|+1)\,c_n(g)\big)\end{aligned}$$

with

$$c_n(g)=\sup_{1\leqq k\leqq n} 1+h(g^{k-1}0,g^k0)\leqq \sup_{1\leqq k\leqq n} 1+\frac{2}{b}\sinh\frac{2}{b}\,d(g^{k-1}0,g^k0)\leqq 1+\frac{2}{b}\sinh\frac{2}{b}\,d(0,g0).$$

It readily follows that $\sum_{n\in\mathbb{Z}^*} e^{-sd(0,g^n0)}\geqq A^{2s}\sum_{n\in\mathbb{Z}^*}\frac{1}{n^{2s}}$ for some positive constant A which implies $\delta_g\geqq 1/2$.

If X is a symmetric space of rank one, the group generated by any parabolic isometry g is of divergence type (see for example [8]); this is also the case if there exists a horoball centered at x_g isometric to a horoball of a symmetric space of rank one. Note that there exist Hadamard manifolds of pinched curvature which possess parabolic isometries of convergent type ([9]).

Theorem III.1. *Let G be a non elementary group of isometries of X. For any $g\in G$ such that* $\sum_{n\in\mathbb{Z}} e^{-\delta_g d(0,g^n0)}=+\infty$, *one has* $\delta_G>\delta_g$.

If G is purely hyperbolic one obtains $\delta_\Gamma>0$; this fact is not new and it holds even if X is a general Gromov hyperbolic space [7]. If G contains parabolic transformations one obtains in particular $\delta_\Gamma>1/2$; this inequality was already proved in constant curvature ([3], [26]).

Proof. One adapts here a Beardon's argument [3]. Fix $g\in G$ such that the series $\sum_{n\in\mathbb{Z}^*} e^{-sd(0,g^n0)}$ diverges at its critical exponent δ_g; since G is non elementary there exists an hyperbolic isometry h such that the group $\langle g, h\rangle$ generated by g and h is an extended Schottky group. One has

$$\begin{aligned}\sum_{\gamma\in\langle g,h\rangle} e^{-sd(0,\gamma 0)} &\geqq \sum_{k\geqq 1}\ \sum_{n_1,\dots,n_k\in\mathbb{N}^*} e^{-sd(0,g^{n_1}hg^{n_2}h\dots g^{n_k}h0)}\\ &\geqq \sum_{k\geqq 1}\ \sum_{n_1,\dots,n_k\in\mathbb{N}^*} e^{-s(d(0,g^{n_1}0)+\cdots+d(0,g^{n_k}0)+kd(0,h0))}\\ &\geqq \sum_{k\geqq 1}\Big(e^{-sd(0,h0)}\sum_{n\geqq 1} e^{-sd(0,g^n0)}\Big)^k.\end{aligned}$$

Since $\lim\limits_{\substack{s\to\delta_g\\ s>\delta_g}} \sum_{n\geqq 1} e^{-sd(0,g^n0)} = +\infty$, one can choose $s>\delta_g$ such that $e^{-sd(0,h0)}\sum_{n\geqq 1} e^{-sd(0,g^n0)}>1$; this implies that the series $\sum_{\gamma\in\langle g,h\rangle} e^{-sd(0,\gamma 0)}$ diverges for some $s>\delta_g$ so that $\delta_G \geqq \delta_{\langle g,h\rangle} > \delta_g$. □

IV. The Patterson-Sullivan measure on Λ

Denote by δ_Γ the exponent of convergence of the Poincaré series $\sum_{\gamma\in\Gamma} e^{-sd(0,\gamma 0)}$ and σ the Patterson-Sullivan measure on Λ [19], [26]. Since the sectional curvature of X is lower bounded, δ_Γ is finite. For any isometry $\gamma\in\Gamma$ one has

$$(*)\qquad \frac{d(\gamma^{-1}\sigma)}{d\sigma}(x) = |\gamma'(x)|^{\delta_\Gamma}\quad \sigma(dx)\text{-a.s.}$$

Recall the *shadow* $\theta(z,r)$ on ∂X of the ball $\mathbb{B}(z,r)$ with center $z\in X$ and radius $r>0$ is the set of points $x\in\partial X$ such that the geodesic ray joining 0 and x meets $\mathbb{B}(z,r)$. Let us recall the following

Lemma IV.1 (Sullivan's shadow lemma [7]). *There exist $C\geqq 1$ and $d_0>0$ such that for every $r\geqq d_0$ and $\gamma\in\Gamma$ one has*

$$\frac{1}{C}e^{-\delta_\Gamma d(0,\gamma 0)} \leqq \sigma\big(\theta(\gamma 0, r)\big) \leqq Ce^{-\delta_\Gamma d(0,\gamma 0)+2r\delta_\Gamma}.$$

When the curvature is constant it is well known that the Patterson-Sullivan measure on Λ has no atom [31]. By the Ping-Pong property and the dynamic of generators on ∂X we show that the same property holds for any extended Schottky group acting on X. The following theorem has been obtained with J. P. Otal.

Theorem IV.2. *The Patterson-Sullivan measure σ associated with Γ has no atomic part.*

Proof. Since Γ is geometrically finite ([5]), Λ is the disjoint union of the radial limit set and the fixed points of parabolic transformations of Γ. Radial limit points cannot be atoms of σ (see for example [33]), thus one just has to check that $\sigma\{x_\alpha\}=0$ for any parabolic generator $\alpha\in\mathscr{A}$.

For $s > \delta_\Gamma$ and $y \in X$ set $g_s(y) = \sum_{\gamma \in \Gamma} e^{-sd(y,\gamma y)} h\big(d(y,\gamma y)\big)$ where h is an increasing function of arbitrary small exponential growth which makes the series $g_s(y)$ diverge at $s = \delta_\Gamma$; more precisely for any $\varepsilon > 0$ there exists $d_\varepsilon > 0$ such that $h(d+t) \leqq e^{\varepsilon t} h(d)$ for any $t \geqq 0$ and $d \geqq d_\varepsilon$. By theorem III.1 one can fix ε such that $\delta_\alpha + \varepsilon < \delta_\Gamma$; consequently, the series $\sum_{n \in \mathbb{Z}} e^{(-\delta_\Gamma + \varepsilon) d(0,\alpha^n 0)}$ converges.

Set $\Gamma' = \{\gamma = a_1 \cdots a_n \in \Gamma \,|\, a_i \neq a_{i+1}^{-1} \text{ and } a_1 \neq \alpha\}$; since the limit points of $\Gamma' y$ belong to $\Lambda - \Lambda_\alpha$ there exists a closed cone C_z of vertex $z \in X$ such that $\Gamma' y \subset C_z$ but $x_\alpha \notin C_z$. Without loss of generality one may suppose that the distance between the origin 0 and the cone C_z is greater than d_ε and that the horosphere centered at x_α containing 0 is included in a cone of vertex 0 which does not intersect C_z; for every $k \geqq 1$ there thus exists a cone C_k of vertex 0 and axis $[0, x_\alpha)$ such that $C_k \cap \alpha^n(C_z) = \emptyset$ when $|n| \leqq k$.

Recall that a Patterson-Sullivan measure σ is a weak limit as $s \to \delta_\Gamma^+$ (along a subsequence if necessary) of the family of measures $\sigma_s = \frac{1}{g_s(y)} \sum_{\gamma \in \Gamma} e^{-sd(0,\gamma y)} h\big(d(0,\gamma y)\big) \delta_{\gamma y}$ where $\delta_{\gamma y}$ is the Dirax mass at γy. By the choice of the cones C_k one has

$$\sigma_s(C_k) \leqq \frac{1}{g_s(y)} \sum_{|n| > k} \sum_{\gamma' \in \Gamma'} e^{-sd(0,\alpha^n \gamma' y)} h\big(d(0,\alpha^n \gamma' y)\big).$$

By hyperbolic geometrical arguments (see for example [8], lemma 3.1) there exists a constant $K > 0$ such that $d(0,\alpha^n 0) + d(0,\gamma' y) - K \leqq d(0,\alpha^n \gamma' y) \leqq d(0,\alpha^n 0) + d(0,\gamma' y)$ for any $n \in \mathbb{Z}^*$ and $\gamma' \in \Gamma'$; consequently, for $s > \delta_\Gamma$ one has, up to multiplicative constants

$$\begin{aligned}\sigma_s(C_k) &\leqq \frac{1}{g_s(y)} \sum_{|n| > k} e^{(-s+\varepsilon) d(0,\alpha^n 0)} \sum_{\gamma' \in \Gamma'} e^{-sd(0,\gamma' y)} h\big(d(0,\gamma' y)\big) \\ &\leqq \Big(\sum_{|n| > k} e^{(-\delta_\Gamma + \varepsilon) d(0,\alpha^n 0)}\Big) \sigma_s(C_z).\end{aligned}$$

Letting $s \to \delta_\Gamma$ one obtains $\sigma\{x_\alpha\} \leqq \sigma(C_k \cap \partial X) \leqq \Big(\sum_{|n| > k} e^{(-\delta_\Gamma + \varepsilon) d(0,\alpha^n 0)}\Big) \sigma(C_z \cap \partial X)$ for any integer $k \geqq 1$. Letting $k \to +\infty$ one obtains $\sigma\{x_\alpha\} = 0$. □

Corollary IV.3. *The group Γ is of divergence type, i.e.* $\sum_{\gamma \in \Gamma} e^{-\delta_\Gamma d(0,\gamma 0)} = +\infty$.

Proof. The following argument is classical (see for example [31] or [33]). Set $\Gamma = \{g, n \geqq 1\}$ and assume $\sum_{n \geqq 1} e^{-\delta_\Gamma d(0, g_n(0))} < +\infty$; the lemma III.1 thus implies $\sum_{n \geqq 1} \sigma\big(\theta(g_n 0, A)\big) < +\infty$ for A large enough. By the Borel-Cantelli lemma one obtains $\sigma\big(\limsup_{n \to +\infty} \theta(g_n 0, A)\big) = 0$. The inclusion $\Lambda^0 \subset \bigcup_{\substack{A \in \mathbb{N} \\ A \geqq d_0}} \bigcap_{N \geqq 1} \bigcup_{n \geqq N} \theta(g_n 0, A)$ leads to $\sigma(\Lambda^0) = 0$ and so $\sigma(\Lambda - \Lambda^0) = 1$; since $\Lambda - \Lambda^0$ is countable, this last equality contradicts the fact that σ has no atomic part. □

When X is the real hyperbolic half space, this divergence property is satisfied for any non elementary geometrically finite discret group [31]; recently this result has been extended by K. Corlette and A. Iozzi [8] to the case where X is a symmetric space of rank one.

V. The geodesic flow and the boundary map

Let $G\Lambda$ be the set of pairs (ξ, x) where ξ is an oriented geodesic on X whose endpoints $\xi^- = \xi(-\infty)$ and $\xi^+ = \xi(+\infty)$ belong to Λ and $x \in \xi$. Denote by 0_ξ the intersection of ξ with the horosphere based at ξ^+ passing through the origin 0 and $\partial^2\Lambda$ the set $\Lambda \times \Lambda$-diagonal. The map $\pi : G\Lambda \to \partial^2\Lambda \times \mathbb{R}$ defined by $\pi(\xi, x) = \big(\xi^-, \xi^+, B_{\xi^+}(0_\xi, x)\big)$ is bijective. The group Γ acts on $\partial^2\Lambda \times \mathbb{R}$ in the following way:

$$\gamma(x_-, x, s) = \big(\gamma(x_-), \gamma(x), s - B_x(0, \gamma^{-1}0)\big)$$

for any $\gamma \in \Gamma$ and $(x_-, x, s) \in \partial^2\Lambda \times \mathbb{R}$. Denote $(g_t)_{t \in \mathbb{R}}$ the geodesic flow on $\partial^2\Lambda \times \mathbb{R}$ defined by $g_t(x_-, x, s) = (x_-, x, s + t)$.

Set $\partial^2\Lambda^0 = \bigcup\limits_{\substack{\alpha, \beta \in \mathscr{A} \\ \alpha \neq \beta}} \Lambda^0_{\alpha^\pm} \times \Lambda^0_{\beta^\pm}$. For any $(x_-, x) \in \partial^2\Lambda^0$ such that a^n is the first term of the sequence $\omega(x)$, set $f(x) = B_x(0, a^n 0)$ and $\bar{T}(x_-, x) = (a^{-n}x_-, a^{-n}x)$; the action of Γ on $\partial^2\Lambda \times \mathbb{R}$ induces a map $\bar{T}_f$ on $\partial^2\Lambda \times \mathbb{R}$ defined by

$$\bar{T}_f(x_-, x, s) = \big(\bar{T}(x_-, x), s - f(s)\big).$$

Remark that $\bar{T}_f$ is invertible with inverse $\bar{T}_f^{-1}(y_-, y, t) = \big(x_-, x, t + f(x)\big)$ where $(y_-, y) = \bar{T}(x_-, x)$. Denote $G\Lambda^0$ the set of pairs $(\xi, x) \in G\Lambda$ such that endpoints of ξ belong to Λ^0. The quotient $G\Lambda^0/\Gamma$ is identified with $\partial^2\Lambda^0 \times \mathbb{R}/\langle \bar{T}_f \rangle$.

Let μ be the measure on $\partial^2\Lambda$ defined by $\mu(dx_-\,dx) = \dfrac{\sigma(dx_-)\sigma(dx)}{D(x_-, x)^{2\delta_\Gamma}}$ and let l be the Lebesgue measure on $\mathbb{R}$; since $D(\Lambda^0_{\alpha^\pm}, \Lambda^0_{\beta^\pm}) > 0$ for every $\alpha, \beta \in \mathscr{A}$, $\alpha \neq \beta$, the restriction μ_0 of μ to the set $\partial^2\Lambda^0$ is finite. Furthermore, the measure $\mu \otimes l$ is invariant under the action of Γ and of the geodesic flow $(g_t)_{t \in \mathbb{R}}$. In paragraph VIII we will prove that $0 < \nu(f) < +\infty$ which readily implies that $\mu_0 \otimes l$ induces on $G\Lambda^0/\Gamma$ a finite measure $\overline{\mu_0 \otimes l}$ invariant under the geodesic flow $(\bar{g}_t)_{t \in \mathbb{R}}$ induced by $(g_t)_{t \in \mathbb{R}}$.

There are close connections between the geodesic flow $(\bar{g}_t)_{t \in \mathbb{R}}$ and the action of Γ on Λ; in particular, if G is a geometrically finite discret group of divergent type, the geodesic flow on the unit tangent bundle of X/Γ is ergodic relatively to $\overline{\mu \otimes l}$ ([18], [31]). In paragraph VI we prove that if Γ is an extended Schottky group then $(\bar{g}_t)_{t \in \mathbb{R}}$ is mixing relatively to $\overline{\mu \otimes l}$; to show this we first have to control the action of Γ on Λ^0.

Let T be the *boundary map* on Λ^0 induced by $\bar{T}$ and defined by $T(x) = a^{-n}x$ where a^n is the first term of $\omega(x)$; this mapping is the geometrical interpretation of the shift operator on the symbolic space $\{\omega(x), x \in \Lambda^0\}$ and its properties will play an important rule in the sequel.

Proposition V.1. *There exists $N \in \mathbb{N}^*$ such that* $\inf_{x \in \Lambda^0} |(T^N)'(x)| > 1.$

Set $B_0 = \inf_{x \in \Lambda^0} |(T^N)'(x)| > 1$; using the mean values relation one obtains the

Corollary V.2. *Let $x, y \in \Lambda^0$ such that the sequences $\omega(x)$ and $\omega(y)$ have the same N first terms. Then $D(T^N x, T^N y) \geqq B_0 D(x, y)$.*

Proof of proposition V.1. Fix $B > 0$ and suppose that for any $n \geqq 0$ there exists $x_n \in \Lambda^0$ such that $|(T^n)'(x_n)| \leqq B$. Set $\omega(x_n) = (a_{nk}^{p_{nk}})_{k \geqq 1}$; without loss of generality one can suppose $a_{n1} = \alpha$, $a_{nn} = \beta_1$ and $a_{n(n+1)} = \beta_2$ for any integer $n \geqq 1$. If $\alpha = \beta_1$, let $a \in \mathscr{A} - \{\beta_1\}$ and set $X_n = a x_{n-1}$ for any $n \geqq 1$; one has $|(T^n)'(X_n)| = |(a^{-1})'(X_n)| |(T^{n-1})'(x_{n-1})|$ and so $|(T^n)'(X_n)| \leqq B \sup_{x \in \Lambda^0} |(a^{-1})'(x)|$ which proves that $(x_n)_{n \geqq 1}$ and $(X_n)_{n \geqq 1}$ satisfy a similar condition. Hence, without loss of generality, one may suppose $\alpha \neq \beta_1^{\pm 1}$.

Set $\gamma_n = a_{n1}^{p_{n1}} \cdots a_{nn}^{p_{nn}}$ and $y_n = \gamma_n^{-1} x_n$. Since $\beta_1 \neq \beta_2$ we have $D(y_n, x_{\gamma_n}^-) \geqq D(C_{\beta_2^\pm}, C_{\beta_1^\pm}) > 0$ so that

$$D(x_n, x_{\gamma_n}^-) = \sqrt{|\gamma_n'(y_n)| |\gamma_n'(x_{\gamma_n}^-)|}\, D(y_n, x_{\gamma_n}^-) \geqq \sqrt{\frac{\Phi(\gamma_n)}{B}}\, D(C_{\beta_2^\pm}, C_{\beta_1^\pm}).$$

Condition $\alpha \neq \beta_1$ implies that the transformations γ_n are not pairwise conjugated; by corollary II.3 one obtains $\lim_{n \to +\infty} \Phi(\gamma_n) = +\infty$ so that $\lim_{n \to +\infty} D(x_n, x_{\gamma_n}^-) = +\infty$ which contradicts the compactness of $(\partial X, D)$. □

Now we construct a T-invariant probability measure on Λ^0. By equality $(*)$ of paragraph IV and by the mean values relation, the measure $\mu_0(dx_- \, dx) = \dfrac{\sigma(dx_-)\sigma(dx)}{D(x_-, x)^{2\delta_\Gamma}}$ on $\partial^2 \Lambda^0$ is $\bar{T}$-invariant. Set $\mu_0(\partial^2 \Lambda^0) = 1/C$ and let $p : \partial^2 \Lambda^0 \to \Lambda^0$ be the projection on the second coordinate; one has

Proposition V.3. *The measure $\nu = Cp(\mu_0)$ is a T-invariant probability measure on Λ^0, absolutely continuous with respect to σ with density h given by*

$$\forall \alpha \in \mathscr{A}, \forall x \in \Lambda^0_{\alpha^\pm} \quad h(x) = C \int_{\Lambda^0 - \Lambda^0_{\alpha^\pm}} \frac{\sigma(dy)}{D(x, y)^{2\delta_\Gamma}}.$$

Let us now introduce the transfer operator P associated with (T, ν). Denote by $\mathbb{L}^1(\Lambda^0, \nu)$ (resp. $\mathbb{L}^\infty(\Lambda^0, \nu)$) the standard completion of the space of Borel functions from Λ^0 into $\mathbb{R}$ which are integrable (resp. bounded) with respect to ν. Since ν is T-invariant, the transformation T induces an isometry on $\mathbb{L}^1(\Lambda^0, \nu)$ defined by $T(\psi) = \psi \circ T$ for any $\psi \in \mathbb{L}^1(\Lambda^0, \nu)$. For any $\varphi \in \mathbb{L}^\infty(\Lambda^0, \nu)$, let $P\varphi$ be the function in $\mathbb{L}^\infty(\Lambda^0, \nu)$ such that

$$\forall \psi \in \mathbb{L}^1(\Lambda^0, \nu) \int_{\Lambda^0} \varphi(x)(T\psi)(x)\nu(dx) = \int_{\Lambda^0} P\varphi(x)\psi(x)\nu(dx).$$

One has $P\varphi(x) = \sum_{y \in \Lambda^0/Ty = x} \frac{h(y)}{h(x)} e^{-\delta_\Gamma f(y)} \varphi(y) = \sum_{\substack{\alpha \in \mathscr{A} \\ n \in \mathbb{Z}^*}} 1_{\Lambda^0 - \Lambda^0_{\alpha^\pm}}(x) \frac{h(\alpha^n x)}{h(x)} |(\alpha^n)'(x)|^{\delta_\Gamma} \varphi(\alpha^n x)$ for ν-almost all x in Λ^0. A priori, P acts on $\mathbb{L}^\infty(\Lambda^0, \nu)$; nevertheless it is possible to define $P\varphi(x)$ in $\overline{\mathbb{R}}^+$ for any positive Borel function φ on Λ in the following way:

Definition V.4. For every Borel function φ from Λ into $\mathbb{R}^+$ and every point $x \in \Lambda$, set

$$P\varphi(x) = \sum_{\substack{\alpha \in \mathscr{A} \\ n \in \mathbb{Z}^*}} p_{\alpha^n}(x) \varphi(\alpha^n x)$$

with $p_{\alpha^n}(x) = 1_{\Lambda - \Lambda_{\alpha^\pm}}(x) \frac{h(\alpha^n x)}{h(x)} |(\alpha^n)'(x)|^{\delta_\Gamma}$.

Note that for every $x \in \Lambda^0$ and every $n \geqq 1$ one has

$$P^n \varphi(x) = \sum_{y \in \Lambda^0/T^n y = x} \frac{h(y)}{h(x)} e^{-\delta_\Gamma S_n f(y)} \varphi(y).$$

In the same way the mapping $\bar{T}_f$ induces a transformation T_f on $\Lambda^0 \times \mathbb{R}$ defined by $T_f(x, s) = \big(Tx, s - f(x)\big)$ for every $(x, s) \in \Lambda^0 \times \mathbb{R}$. In some sense, the set $\partial^2 \Lambda^0$ is a section for the geodesic flow on $\partial^2 \Lambda \times \mathbb{R}/\Gamma$ and T_f is the first return map for this flow on this section; the transformation T_f memorizes the "travel time" between two consecutive passages through $\partial^2 \Lambda^0$. Let us introduce the operator $\tilde{P}$ associated with T_f:

Definition V.5. For every Borel function ψ from $\Lambda \times \mathbb{R}$ into $\mathbb{R}^+$ and every $(x, t) \in \Lambda \times \mathbb{R}$ set

$$\tilde{P}\psi(x, t) = \sum_{\substack{\alpha \in \mathscr{A} \\ n \in \mathbb{Z}^*}} p_{\alpha^n}(x) \psi\big(\alpha^n x, t + f(\alpha^n x)\big).$$

Note that for every $(x, t) \in \Lambda^0 \times \mathbb{R}$ one has

$$\tilde{P}\psi(x, t) = \sum_{y \in \Lambda^0/Ty = x} \frac{h(y)}{h(x)} e^{-\delta_\Gamma f(y)} \psi\big(y, t + f(y)\big)$$

and

$$\forall n \geqq 1 \quad \tilde{P}^n \psi(x, t) = \sum_{y \in \Lambda^0/T^n y = x} \frac{h(y)}{h(x)} e^{-\delta_\Gamma S_n f(y)} \psi\big(y, t + S_n f(y)\big).$$

VI. A renewal theorem to prove that the geodesic flow is mixing

Notation. From this section on, Γ is an extended Schottky group and δ is the exponent of convergence of the Poincaré series $\sum_{\gamma \in \Gamma} e^{-sd(0, \gamma 0)}$.

We state here a classical renewal theorem which describes the behaviour as a goes to $\pm\infty$ of the potential $\sum_{n=0}^{+\infty} \tilde{P}^n\big((x, a), dy\, dt\big)$ and we show how one may deduce the mixing property of the geodesic flow $(\bar{g}_t)_{t \in \mathbb{R}}$. First let us introduce a functional space L on which P acts.

Notation. Let L be the space of functions φ from Λ into $\mathbb{C}$ such that

$$\|\varphi\| = |\varphi|_\infty + m(\varphi) < +\infty$$

where $|\cdot|_\infty$ is the norm of uniform convergence on Λ and

$$m(\varphi) = \sup_{\alpha \in \mathscr{A}} \sup_{\substack{x, y \in \Lambda_{\alpha^\pm} \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{D(x, y)^{\delta_0}} \quad \text{with } \delta_0 = \inf\{1, \delta\}.$$

For every $\lambda \in \mathbb{R}$ let P_λ be the operator defined by $P_\lambda \varphi = P(e^{i\lambda f}\varphi)$. In paragraph VIII we will prove the following facts.

Properties VI.1 (properties R)**.**

(R1) *The operator P acts on $(L, \|\,.\,\|)$.*

(R2) *One has $0 < \nu(f) < +\infty$ and $\sup_{x \in X} Pf^n(x) < +\infty$ for any $n \geqq 1$.*

(R3) *For any real number λ the operator P_λ acts on L; moreover, the mapping $\lambda \mapsto P_\lambda$ is analytic from $(\mathbb{R}, |\,.\,|)$ into the Banach space $(\mathscr{L}(L), \|\,.\,\|_{\mathscr{L}(L)})$ of continuous linear applications on $(L, \|\,.\,\|)$ with the usual norm.*

(R4) *One has $P1 = 1$, the eigenvalue 1 is simple and isolated in the spectrum of P and ν is the projection on the associated eigenspace $\mathbb{C}\,1_\Lambda$.*

(R5) *For every $\lambda \neq 0$ the spectral radius of P_λ on $(L, \|\,.\,\|)$ is strictly less than 1.*

Note that property (R5) is closely related to the fact that Γ contains parabolic transformations; if Γ is a Schottky group this property is not proved.

Using arguments developped in [1], [14] one proves the following theorem:

A renewal theorem. *Assume that (P, f) satisfies properties R. Then for any compact set $K \subset \Lambda \times \mathbb{R}$, for any bounded Borel function $\varphi : \Lambda \to \mathbb{R}$ whose discontinuity points are a ν-negligeable set and for any continuous function $u : \mathbb{R} \to \mathbb{R}$ with compact support one has*

$$\lim_{a \to +\infty} \sup_{(x,s) \in K} \left| \sum_{n=0}^{+\infty} \tilde{P}^n(\varphi \otimes u)(x, s-a) - \frac{\nu(\varphi) \int_{\mathbb{R}} u(t)\,dt}{\nu(f)} \right| = 0$$

and

$$\lim_{a \to +\infty} \sup_{(x,s) \in K} \left| \sum_{n=0}^{+\infty} \tilde{P}^n(\varphi \otimes u)(x, s+a) \right| = 0.$$

Throughout this paragraph we will assume that properties R hold. Recall that the inequality $0 < \nu(f) < +\infty$ implies that $\overline{\mu \otimes l}$ is finite on $G\Lambda/\Gamma$ and that $\overline{\mu \otimes l}(G\Lambda/\Gamma) = \nu(f)$.

Theorem VI.2. *Let Γ be an extended Schottky group. Then the geodesic flow $(\bar{g}_t)_{t\in\mathbb{R}}$ on $G\Lambda/\Gamma$ is mixing relatively to $\overline{\mu \otimes l}$.*

Proof. We adapt here Y. Guivarc'h and J. Hardy's proof of the mixing property for a special flow constructed with a Hölder continuous function over a subshift of finite type [14]. One has to show that for every functions Φ and Ψ in $\mathbb{L}^2(G\Lambda/\Gamma, \overline{\mu \otimes l})$

$$\lim_{t\to+\infty} I_t(\Phi, \Psi) = \frac{1}{\nu(f)} \overline{\mu \otimes l}(\Phi)\overline{\mu \otimes l}(\Psi)$$

with $I_t(\Phi, \Psi) = \int\limits_{\partial^2\Lambda\times\mathbb{R}/\Gamma} \Phi(x_-, x, s)\, \Psi \circ \bar{g}_t(x_-, x, s)\, \overline{\mu \otimes l}(dx_-\, dx\, ds)$. By proposition III.2 the measure σ has no atomic part; the same holds for $\mu \otimes l$ and so, without loss of generality, one may suppose that Φ and Ψ are defined on $G\Lambda^0$; one identifies $G\Lambda^0/\Gamma$ with a suitable fundamental domain $S \subset \partial^2\Lambda^0 \times \mathbb{R}$ for the action of the group $\langle \bar{T}_f \rangle$ and we also denote by $(\bar{g}_t)_{t\in\mathbb{R}}$ the corresponding flow on S. Using a density argument consider $\Phi = \varphi \otimes u$ where φ is Hölder continuous on $\partial^2\Lambda^0$, u is continuous on $\mathbb{R}$ and the support of Φ is included in S. In the same way, it suffices to consider a Hölder continuous function ψ on $\partial^2\Lambda^0$ and a continuous function on $\mathbb{R}$ whose supports are compact and such that

$$\forall (x_-, x, s) \in S \quad \Psi(x_-, x, s) = \sum_{n\in\mathbb{Z}} \psi \otimes v\big(\bar{T}_f^n(x_-, x, s)\big).$$

One thus obtains $I_t(\Phi, \Psi) = I_t^+(\Phi, \Psi) + I_t^-(\Phi, \Psi)$ with

$$I_t^+(\Phi, \Psi) = \sum_{n\geqq 0} \int\limits_{\Lambda^0\times\mathbb{R}} \varphi(x_-, x)\, u(s)\, \psi \otimes v\big(\bar{T}_f^n(x_-, x, s+t)\big)\, \mu(dx_-\, dx)\, ds$$

and

$$I_t^-(\Phi, \Psi) = \sum_{n\geqq 1} \int\limits_{\Lambda^0\times\mathbb{R}} \varphi(x_-, x)\, u(s)\, \psi \otimes v\big(\bar{T}_f^{-n}(x_-, x, s+t)\big)\, \mu(dx_-\, dx)\, ds.$$

Let us first deal with the term $I_t^+(\Phi, \Psi)$. Identify (x_-, x) with a bilateral sequence $(a_i^{p_i})_{i\in\mathbb{Z}}$ with $\omega(x) = (a_i^{p_i})_{i\geq 1}$ and $\omega(x_-) = (a_i^{-p_i})_{i\leq 0}$. Using again a density argument, one may suppose that φ and ψ do only depend on the unilateral sequence $(a_i^{p_i})_{i\geqq -l}$ with $l \geqq 0$; moreover, one may suppose $l = 0$ because the measure μ is $\bar{T}$-invariant. Then, for a suitable choice of $\Phi = \varphi \otimes u$ and $\Psi = \psi \otimes v$ one has

$$\begin{aligned} I_t^+(\Phi, \Psi) &= \sum_{n=0}^{+\infty} \int\limits_{\Lambda^0\times\mathbb{R}} \frac{\varphi(x)}{h(x)}\, u(s)\, \psi\big(T^n(x)\big)\, v\big(s + t - S_n f(x)\big)\, \nu(dx)\, ds \\ &= \sum_{n=0}^{+\infty} \int\limits_{\Lambda^0\times\mathbb{R}} \tilde{P}^n\left(\frac{\varphi}{h} \otimes u\right)(x, s-t)\, \psi(x)\, v(s)\, \nu(dx)\, ds. \end{aligned}$$

On the other hand, since $\mu \otimes l$ is $\bar{T}_f$-invariant, one has

$$I_t^-(\Phi,\Psi) = \sum_{n=1}^{+\infty} \int_{\Lambda^0\times\mathbb{R}} \varphi \otimes u\big(\bar{T}_f^n(x_-,x,s)\big)\,\psi(x_-,x)\,v(s+t)\,\mu(dx_-\,dx)\,ds$$

and for a suitable choice of functions Φ and Ψ one has

$$I_t^-(\Phi,\Psi) = \sum_{n=1}^{+\infty} \int_{\Lambda^0\times\mathbb{R}} \varphi(x)\,u(s)\,\tilde{P}^n\left(\frac{\psi}{h}\otimes v\right)(x,s+t)\,v(dx)\,ds\,.$$

By the renewal theorem one obtains

$$\lim_{t\to+\infty} \sup_{(x,s)\in\mathrm{Supp}(\Psi)} \left| \sum_{n=0}^{+\infty} \tilde{P}^n\left(\frac{\varphi}{h}\otimes u\right)(x,s-t) - \frac{v(\varphi)\int_{\mathbb{R}} u(y)\,dy}{v(f)} \right| = 0$$

and

$$\lim_{t\to+\infty} \sum_{n=1}^{+\infty} \tilde{P}^n(\psi\otimes v)(x,s+t) = 0\,.$$

The Lebesgue dominated convergence theorem allows us to conclude

$$\lim_{t\to+\infty} I_t(\Phi,\Psi) = \frac{1}{v(f)}\,\overline{\mu\otimes l}(\Phi)\,\overline{\mu\otimes l}(\Psi)$$

which finishes the proof. □

VII. An harmonic renewal theorem to count closed geodesics on X/Γ

Consider the equivalence relation $\sim$ on Γ defined by $\gamma_1 \sim \gamma_2$ if and only if γ_1 and γ_2 are conjugated in Γ. For any class $c \in \Gamma/\sim$ choose $\gamma_c = a_1^{n_1}\cdots a_k^{n_k}$ in c such that $a_i \in \mathscr{A}$, $a_{i+1} \neq a_i$ for $1 \leqq i < k$ and $a_1 \neq a_k$. Denote by $\mathscr{C}_0$ the set of γ_c which are primitive (i.e. $\gamma_c \neq \gamma^n$ for all $\gamma \in \Gamma$) and hyperbolic. Let $\gamma_0 = a_1^{n_1}\cdots a_k^{n_k} \in \mathscr{C}_0$; the expansion $\omega(x_{\gamma_0})$ of the attractive fixed point x_{γ_0} of γ_0 is periodic with period $a_1^{n_1},\ldots,a_k^{n_k}$. Set $l(\gamma_0) = d\big(z,\gamma_0(z)\big)$ where z belongs to the axis of γ_0; one has $l(\gamma_0) - B_{x_{\gamma_0}}\big(0,\gamma_0(0)\big)$. Recall that $T:\Lambda^0 \to \Lambda^0$ and $f:\Lambda^0 \to \mathbb{R}$ are defined by $T(x) = a^{-n}(x)$ and $f(x) = B_x(0,a^n 0)$ where a^n is the first term of $\omega(x)$. For any $k \geqq 1$ set $S_k f(x) = f(x) + \cdots + f(T^{k-1}x)$. Using the fact that $B_{\gamma x}(\gamma z_1,\gamma z_2) = B_x(z_1,z_2)$ for every isometry γ and that $B_x(z_1,z_2) = B_x(z_1,z) + B_x(z,z_2)$ one obtains $l(\gamma_0) = S_k f(x_{\gamma_0}^+)$ for any $\gamma_0 = a_1^{n_1}\cdots a_k^{n_k} \in \mathscr{C}_0$.

Conversely, consider a T-periodic point $x \in \partial X$ and let k be its least period; one has $\omega(x) = a_1^{n_1},\ldots,a_k^{n_k},a_1^{n_1},\ldots,a_k^{n_k},\ldots$. Set $\gamma = a_1^{n_1}\cdots a_k^{n_k}$; since $a_k \neq a_1$ we have $\lim_{p\to+\infty} \gamma^p 0 = x$ which shows that γ is hyperbolic and x is its attractive fixed point. Moreover $\gamma \in \mathscr{C}_0$ since k is the least period of x.

Finally, the number $\pi(a)$ of γ in $\mathscr{C}_0 - \{\alpha_1,\ldots,\alpha_{N_1}\}$ such that $l(\gamma) \leqq a$ is given by

$$\pi(a) = \sum_{k=1}^{+\infty} \frac{1}{k}\,\#\{x \in \Lambda^0 \,|\, x \text{ is } T \text{ periodic, } k \text{ is the least period of } x \text{ and } S_k f(x) \leqq a\}\,.$$

By corollary II.3 the number $\pi(a)$ is finite for every $a > 0$. In this paragraph we prove the

Theorem VII.1. *Let Γ be an extended Schottky group and δ be the exponent of convergence of the Poincaré series associated with Γ. Then, the function $a \to \pi(a)$ is equivalent to $\frac{e^{a\delta}}{a\delta}$ when a goes to $+\infty$.*

To prove this theorem, one approximates $\pi(a)$ by harmonic potentials $\sum_{n \geqq 1} \frac{1}{n} \tilde{P}^n$; in paragraph VIII we prove that (P, f) satisfies Properties R given in the previous section which allows us to state the following result.

A harmonic renewal theorem. *For any bounded Borel function $\varphi : \Lambda \to \mathbb{R}$ whose discontinuity points form a ν-negligeable set in Λ and for any continuous function $u : \mathbb{R} \to \mathbb{R}$ with compact support one has*

$$\lim_{a \to +\infty} \sup_{x \in \Lambda} \left| a \sum_{n=1}^{+\infty} \frac{1}{n} \tilde{P}^n (\varphi \otimes u)(x, -a) - \nu(\varphi) \int_{\mathbb{R}} u(t)\, dt \right| = 0\,.$$

In particular, if $\nu(\varphi) > 0$, and if u is a non decreasing continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$ one has

$$\lim_{a \to +\infty} \sup_{x \in \Lambda} \left| a \frac{\sum_{n=1}^{+\infty} \frac{1}{n} \tilde{P}^n (\varphi \otimes u 1_{[0,a]})(x, 0)}{\frac{\nu(\varphi)}{a} \int_0^a u(t)\, dt} - 1 \right| = 0\,.$$

Notation. Let $a > 0$ and A be a Borel set included in Λ. Set $g_A = \frac{1_A}{h}$ and $\xi_a(t) = e^{\delta t} 1_{[0,a]}(t)$. For any $(x, t) \in \Lambda \times \mathbb{R}$ set $N_{(x,a)}(A) = \sum_{n=1}^{+\infty} \frac{1}{n} \tilde{P}^n (g_A \otimes \xi_a)(x, 0)$.

Applying the above harmonic renewal theorem with $\varphi = g_A$ and $u 1_{[0,a]} = \xi_a$ one thus obtains the following

Corollary VII.2. *For every Borel set $A \subset \Lambda$ such that $\sigma(A) > 0$ and with ν-negligeable boundary, $N_{(x,a)}(A)$ is equivalent to $h(x) \frac{e^{a\delta}}{a\delta} \sigma(A)$ as a goes to $+\infty$, uniformly in $x \in \Lambda$.*

Proof of theorem VII.1. Set

$$N(a) = \sum_{n=1}^{+\infty} \frac{1}{n} \# \{ x \in \Lambda^0 \,|\, T^n x = x \text{ and } S_n f(x) \leqq a \}\,.$$

One has $N(a) - N(a/2) \leqq \pi(a) \leqq N(a)$; to prove theorem VII.1 it suffices to show that $\lim_{a \to +\infty} \frac{N(a)}{e^{a\delta}/a\delta} = 1$. The following proposition is proved at the end of the current paragraph.

Proposition VII.3 (perturbation of T-periodic points). *There exists $k_0 \in \mathbb{N}^*$ such that for any $k \geqq k_0$ there exist a countable partition $(\Lambda^0_{ki})_{i \geqq 1}$ of open sets in Λ, a sequence of positive constants $(C_{ki})_{i \geqq 1}$ with $\sum_{i \geqq 1} C_{ki} < +\infty$ and $\theta_k \in \mathbb{R}^{*+}$ with $\lim_{k \to +\infty} \theta_k = 0$ such that*

(i) $\lim_{k \to +\infty} \sigma(\Lambda^0_{ki}) = 0,$

(ii) *for any x_{ki} in (Λ^0_{ki}) one has*

$$\sum_{i=1}^{+\infty} h(x_{ki}) N_{(x_{ki}, a-\theta_k)}(\Lambda^0_{ki}) \leqq N(a) \leqq \sum_{i=1}^{+\infty} h(x_{ki}) N_{(x_{ki}, a+\theta_k)}(\Lambda^0_{ki}),$$

(iii) $N_{(x_{ki}, a)}(\Lambda^0_{ki}) \leqq C_{ki} \dfrac{e^{a\delta}}{a\delta}$ *for any $a > 0$.*

Applying Fatou's lemma in the left inequality (ii) of this proposition one has by corollary VII.2

$$e^{-\delta\theta_k} \sum_{i=1}^{+\infty} h(x_{ki}) \sigma(\Lambda^0_{ki}) \leqq \liminf_{a \to +\infty} \frac{a\delta}{e^{a\delta}} N(a).$$

In the same way, Lebesgue dominated convergence theorem and inequality (iii) give

$$\limsup_{a \to +\infty} \frac{a\delta}{e^{a\delta}} N(a) \leqq e^{\delta\theta_k} \sum_{i=1}^{+\infty} h(x_{k,i}) \sigma(\Lambda^0_{k,i})$$

so that

$$e^{-\delta\theta_k} \sum_{i=1}^{+\infty} h(x_{ki}) \sigma(\Lambda^0_{ki}) \leqq \liminf_{a \to +\infty} \frac{a\delta}{e^{a\delta}} N(a) \leqq \limsup_{a \to +\infty} \frac{a\delta}{e^{a\delta}} N(a) \leqq e^{\delta\theta_k} \sum_{i=1}^{+\infty} h(x_{ki}) \sigma(\Lambda^0_{ki}).$$

Since $\sigma(\Lambda - \Lambda^0) = 0$ and $\lim_{k \to +\infty} \sigma(\Lambda^0_{ki}) = 0$ one has $\lim_{k \to +\infty} \sum_{i=1}^{+\infty} h(x_{ki}) \sigma(\Lambda^0_{ki}) = \int_\Lambda h(x) \sigma(dx) = 1$. Letting $k \to +\infty$ one obtains $\lim_{a \to +\infty} \dfrac{a\delta}{e^{a\delta}} N(a) = 1$. □

Proof of proposition VII.3. Let $k \in \mathbb{N}^*$ and consider the equivalence relation $\mathscr{R}_k$ on Λ^0 defined by $x \mathscr{R}_k y$ if and only if $\omega(x)$ and $\omega(y)$ have the same k first terms. This relation induces a partition $(\Lambda^0_{ki})_{i \geqq 1}$ on Λ^0.

(i) Let us first prove that $\lim_{k \to +\infty} \sigma(\Lambda^0_{ki}) = 0$. Fix $i \geqq 1$ and for any $x \in \Lambda^0_{ki}$ set $\omega(x) = (a^{p_{kj}}_{kj})_{j \geqq 1}$ and $\gamma_{kj} = a^{p_{k1}}_{k1} \cdots a^{p_{kk}}_{kk}$; one has $\sigma(\Lambda^0_{ki}) = \int_{\Lambda^0 - \Lambda^0_{a_{kk}}} |(\gamma'_{kk})(y)|^{2\delta} \sigma(dy)$. Without loss of generality one may suppose $a_{k1} \neq a^{\pm 1}_{kk}$ for any $k \geqq 1$ (if this condition does not hold it suffices to replace γ_{kk} with $a_1 \gamma_{kk}$ where $a_1 \neq a^{\pm 1}_{kk}$). Then γ_{kk} is hyperbolic and $\lim_{k \to +\infty} \Phi(\gamma_{kk}) = +\infty$ by corollary II.3. Consequently

$$|\gamma'_{kk}(y)| = \frac{D^2(\gamma_{kk} y, x^-_{\gamma_{kk}})}{\Phi(\gamma_{kk}) D^2(y, x^-_{\gamma_{kk}})} \geqq \frac{\|D\|^2_\infty}{\Phi(\gamma_{kk}) D^2(\Lambda_{a_{kk}}, \Lambda_{a_{kk}})}$$

so that $\lim_{k \to +\infty} \sigma(\Lambda^0_{ki}) = 0$.

(ii) For every $i \geqq 1$ fix $x_{ki} \in \Lambda^0_{ki}$. Let $x \in \Lambda^0_{ki}$ such that $T^n x = x$ for some $n \geqq 1$, denote by $\tilde{x}$ the unique point of Λ^0_{ki} such that $T^n \tilde{x} = x_i$ and $\omega(x)$, $\omega(\tilde{x})$ have the same n first terms. There is a bijection between the sets $\{x \in \Lambda^0 \mid T^n x = x\} \cap \Lambda^0_{ki}$ and $T^{-n}(\{x_{ki}\}) \cap \Lambda^0_{ki}$; the following lemma whose proof is given further allows us to control the difference $S_n f(x) - S_n f(\tilde{x})$.

Lemma VII.4. *There exist $k_0 \in \mathbb{N}^*$, $A, B \in \mathbb{R}^{*+}$ and $0 < r < 1$ such that for any $l \geqq k_0$ and any $x, y \in \Lambda^0$ whose associated sequences $\omega(x)$ and $\omega(y)$ have the same l first terms, one has $|f(x) - f(y)| \leqq AD(Tx, Ty) \leqq Br^l$.*

Now fix $k \geqq k_0$; if $T^n x = x$ then $\omega(x)$ and $\omega(\tilde{x})$ have the same $n + k$ first terms so that $|S_n f(x) - S_n f(\tilde{x})| \leqq \theta_k$ with $\theta_k = B \sum_{l=k}^{+\infty} r^l$; inequalities (ii) of proposition VII.3 follow immediately.

(iii) For any Borel set $A \subset \Lambda^0$ set $N_{(x_{ki}, a)}(A) = N'_{(x_{ki}, a)}(A) + N''_{(x_{ki}, a)}(A)$ with

$$N'_{(x_{ki}, a)}(A) = \sum_{n=1}^{k} \frac{1}{n} \sum_{y/T^n y = x_{ki}} 1_A(y) 1_{[0,a]}(S_n f(y))$$

and

$$N''_{(x_{ki}, a)}(A) = \sum_{n=k+1}^{+\infty} \frac{1}{n} \sum_{y/T^n y = x_{ki}} 1_A(y) 1_{[0,a]}(S_n f(y)).$$

By lemmas I.3 and I.4 there exist $C > 1$ and sequences $(K_{\alpha^n})_{n \in \mathbb{Z}^*}$, $\alpha \in \mathscr{A}$ such that for any $y \in \Lambda^0 - \Lambda^0_\alpha$ one has $|(\alpha^n)'(y)| \leqq CK_{\alpha^n}$. Fix $i \geqq 1$ and let $a_1^{p_1}, \ldots, a_k^{p_k}$ be the k first terms of $\omega(x_{ki})$; for any $1 \leqq n \leqq k$ there is a unique $y \in \Lambda^0_{ki}$ such that $T^n y = x_{ki}$ and one has $S_n f(y) \geqq |\mathrm{Log}\, K_{a_1^{p_1}} \cdots K_{a_n^{p_n}}| - n \,\mathrm{Log}\, C$. So

$$N'_{(x_{ki}, a)}(\Lambda^0_{ki}) = \sum_{n=1}^{k} \frac{1}{n} 1_{[0, a + n \mathrm{Log} C]}(|\mathrm{Log}\, K_{a_1^{p_1}} \cdots K_{a_n^{p_n}}|).$$

Since $x \mapsto \dfrac{e^x}{1+x}$ is increasing on $\mathbb{R}^+$, for any $b > 0$ we have $1_{[0,b]}(x) \leqq \dfrac{e^{b\delta}}{1 + b\delta} \dfrac{1 + x\delta}{e^{x\delta}}$ and so $N'_{(x_{ki}, a)}(\Lambda^0_{ki}) \leqq C'_i \dfrac{e^{a\delta}}{a\delta}$ with $C'_i = \sum_{n=1}^{k} \dfrac{C^{n\delta}}{n} (1 + \delta |\mathrm{Log}\, K_{a_1^{p_1}} \cdots K_{a_n^{p_n}}|)(K_{a_1^{p_1}} \cdots K_{a_n^{p_n}})^\delta$.

On the other hand

$$N''_{(x_{ki}, a)}(\Lambda^0_{ki}) = \sum_{n=k+1}^{+\infty} \frac{1}{n} \sum_{z/T^{n-k} z = x_{ki}} \sum_{y/T^k y = z} 1_{\Lambda^0_{ki}}(y) 1_{[0,a]}(S_{n-k} f(z) + S_k f(y)).$$

If $y \in \Lambda^0$, the k first terms of $\omega(y)$ are the ones of $\omega(x_{ki})$ and so $S_k f(y) \geqq A_{ki}$ with $A_{ki} = \mathrm{Log}\, K_{a_1^{p_1}} \cdots K_{a_k^{p_k}}$. We thus have $N''_{(x_{ki}, a)}(\Lambda^0_{ki}) \leqq N_{(x_{ki}, a - A_{ki})}(\Lambda^0)$; so $N''_{(x_{ki}, a - A_{ki})}(\Lambda^0) = 0$ if $a \leqq A_{ki}$ and, by corollary VII.2, $N''_{(x_{ki}, a)}(\Lambda^0_{ki}) \leqq C'' \dfrac{e^{\delta(a - A_{ki})}}{1 + \delta(a - A_{ki})}$ if $a > A_{ki}$. Finally $N''_{(x_{ki}, a)}(\Lambda^0_{ki}) \leqq C''_i \dfrac{e^{a\delta}}{a\delta}$ with $C''_i = C'' e^{-\delta A_{ki}}$.

By the following lemma, the sums

$$\sum_{\substack{a_1,\dots,a_k\in\mathscr{A}\\ a_{i+1}\neq a_i}}\ \sum_{p_1,\dots,p_k\in\mathbb{Z}^*}(K_{a_1^{p_1}}\cdots K_{a_n^{p_n}})^\delta \text{ and } \sum_{\substack{a_1,\dots,a_k\in\mathscr{A}\\ a_{i+1}\neq a_i}}\ \sum_{p_1,\dots,p_k\in\mathbb{Z}^*}|\mathrm{Log}\,K_{a_1^{p_1}}\cdots K_{a_n^{p_n}}|(K_{a_1^{p_1}}\cdots K_{a_n^{p_n}})^\delta$$

are finite so that $\sum_{i\geqq 1} C_i'$ and $\sum_{i\geqq 1} C_i''$ converge. □

Lemma VII.5. *There exists* $\varepsilon>0$ *such that* $\sum_{n\in\mathbb{Z}^*} K_{\alpha^n}^{\delta-\varepsilon}<+\infty$ *for every* $\alpha\in\mathscr{A}$.

Proof of lemma VII.4. Fix x, y in Λ^0 such that the sequences $\omega(x)$ and $\omega(y)$ have the same l first terms. By proposition V.1, there exist $N\geqq 1$ and $B_0>1$ such that $D(T^N x, T^N y)\geqq B_0 D(x,y)$; consequently $D(x,y)\leqq K_1 r^l$ for some $0<r<1$ and $K_1>0$.

Now let $\alpha\in\mathscr{A}$ and $n\in\mathbb{Z}^*$ such that α^n is the first term of the sequences $\omega(x)$ and $\omega(y)$; $T(x)$ and $T(y)$ do not belong to Λ_α and by lemmas I.3 and I.4 (applied with $\gamma=\alpha$ and $E=\Lambda-\Lambda_{\alpha^\pm}$) there exists $K_2>0$ such that

$$\frac{\|(\alpha^n)'(Tx)|-|(\alpha^n)'(Ty)\|}{|(\alpha^n)'(Ty)|}\leqq K_2 D(Tx,Ty)\,.$$

One concludes using the local expansion $|\mathrm{Log}(1+u)|\leqq 2|u|$ for $|u|$ small enough. □

Proof of lemma VII.5. Fix $\alpha\in\mathscr{A}$; by lemmas I.3 and I.4, there exist $A_\alpha\geqq 1$ and $(K_{\alpha^n})_{n\in\mathbb{Z}^*}$ such that

$$\forall x\in\Lambda-\Lambda_{\alpha^\pm}\quad \frac{K_{\alpha^n}}{A_\alpha}\leqq|(\alpha^n)'(x)|\leqq A_\alpha K_{\alpha^n}\,.$$

Let β be in $\mathscr{A}-\{\alpha\}$; one has $\sum_{n\in\mathbb{Z}^*}\frac{\sigma(\Lambda_\beta)}{A_\alpha^\delta}K_{\alpha^n}^\delta\leqq\sigma\big(\bigcup_{n\in\mathbb{Z}^*}\alpha^n(\Lambda_\beta)\big)<+\infty$ so that the series $\sum_{n\in\mathbb{Z}^*}K_{\alpha^n}^\delta$ converges. The same argument holds if one replaces Γ with a subgroup Γ_1 containing α^k and β^k for some $k\geqq 1$ and whose exponent of convergence is strictly less than δ (this is possible by the critical gap property). This readily ensures that the sum $\sum_{n\in\mathbb{Z}^*}K_{\alpha^n}^{\delta-\varepsilon}$ is finite for some $\varepsilon>0$ small enough. □

VIII. Spectrum of Fourier operators P_λ, $\lambda\in\mathbb{R}$

Recall that P is the transfer operator associated with (T,ν); for every Borel function φ from Λ into $\mathbb{R}^+$ one has

$$\forall x\in\Lambda\quad P\varphi(x)=\sum_{\alpha\in\mathscr{A}}\sum_{n\in\mathbb{Z}^*}p_{\alpha^n}(x)\,\varphi\big(\alpha^n(x)\big)$$

with $p_{\alpha^n}(x)=1_{\Lambda-\Lambda_{\alpha^\pm}}(x)\,\frac{h(\alpha^n x)}{h(x)}\,|(\alpha^n)'(x)|^\delta$ and $h(x)=\int_{\Lambda-\Lambda_{\alpha^\pm}}\frac{\sigma(dy)}{D(x,y)^{2\delta}}$ for every $x\in\Lambda_{\alpha^\pm}$. Furthermore, for every $\lambda\in\mathbb{R}$ one has $P_\lambda(\varphi)=P(e^{i\lambda f}\varphi)$.

We consider the space L of functions φ from Λ into $\mathbb{C}$ such that $\|\varphi\|=|_\infty+m(\varphi)<+\infty$ where $|.|_\infty$ is the norm of uniform convergence on Λ and $m(\varphi)=\sup_{\alpha\in\mathscr{A}}\sup_{\substack{x,y\in\Lambda_\alpha\\ x\neq y}}\frac{|\varphi(x)-\varphi(y)|}{D(x,y)^{\delta_0}}$ with $\delta_0=\inf\{1,\delta\}$. We have $1\in L$, $\forall\varphi\in L\quad|\varphi|_\infty\leqq\|\varphi\|$

and L is dense in the space of continuous functions on Λ normed with $|\,.\,|_\infty$. Moreover, $(L, \|\,.\,\|)$ is a Banach space and, by Ascoli's theorem, the canonical one-to-one map from $(L, \|\,.\,\|)$ into $(L, |\,.\,|_\infty)$ is compact.

In the present paragraph we will use intensively the following result which is a consequence of lemmas I.3, I.4 and VII.5.

Lemma VIII.1 (dynamic of generators). *For every $\alpha \in \mathscr{A}$ there exists $A_\alpha \geqq 1$ and a sequence $(K_{\alpha^n})_{n \in \mathbb{Z}^*}$ such that*

(1) $\forall x \in \Lambda - \Lambda_{\alpha^\pm}, \forall n \in \mathbb{Z}^* \quad \dfrac{K_{\alpha^n}}{A_\alpha} \leqq |(\alpha^n)'(x)| \leqq A_\alpha K_{\alpha^n}$,

(2) $\forall x, y \in \Lambda - \Lambda_{\alpha^\pm}, \forall n \in \mathbb{Z}^* \quad \| (\alpha^n)'(x)| - |(\alpha^n)'(y)\| \leqq A_\alpha K_{\alpha^n} D(x, y)$.

Moreover if α is hyperbolic one has $K_{\alpha^n} = 1/\Phi(\alpha)^{|n|}$ with $\Phi(\alpha) > 1$ and if α is parabolic one has $\lim_{n \to +\infty} K_{\alpha^n}^{1/n} = 1$ and $\sum_{n \in \mathbb{Z}^*} K_{\alpha^n}^{\delta - \varepsilon} < +\infty$ for some $\varepsilon > 0$.

Let us now show that properties VI.1 (properties R) hold.

Property (R1).

Proposition VIII.2. *The operator P acts on L.*

Proof. Since $D(\Lambda_{\alpha^\pm}, \Lambda_{\beta^\pm}) > 0$ for $\alpha, \beta \in \mathscr{A}, \alpha \neq \beta$, the function h belongs to L and is non negative. Proposition VIII.2 is thus a direct consequence of the following result:

Lemma VIII.3. *For every $\alpha \in \mathscr{A}$ and every $n \in \mathbb{Z}^*$ the function p_{α^n} belongs to L; furthermore, the sequence $\left(\dfrac{\|p_{\alpha^n}\|}{K_{\alpha^n}^\delta}\right)_{n \in \mathbb{Z}^*}$ is bounded.*

Proof. By lemma VIII.1, there exists $A > 0$ such that for every $n \in \mathbb{Z}^*$ and every $\alpha \in \mathscr{A}$ one has $|(\alpha^n)'(x)| \leqq A K_{\alpha^n}$ and $\big||(\alpha^n)'(x)| - |(\alpha^n)'(y)|\big| \leqq A K_{\alpha^n} D(x, y)$ for every $x, y \in \Lambda - \Lambda_{\alpha^\pm}$; this readily implies $D(\alpha^n x, \alpha^n y) \leqq A K_{\alpha^n} D(x, y)$. Consequently $\|p_{\alpha^n}\|_\infty \leqq A |h|_\infty \left|\dfrac{1}{h}\right|_\infty K_{\alpha^n}^\delta$ and for any $x, y \in \Lambda - \Lambda_{\alpha^\pm}$ we have

$$\begin{aligned} |p_{\alpha^n}(x) - p_{\alpha^n}(y)| &\leqq \frac{|h(\alpha^n x) - h(\alpha^n y)|}{h(x)} |(\alpha^n)'(x)|^\delta + \left|\frac{1}{h(x)} - \frac{1}{h(y)}\right| h(\alpha^n y) |(\alpha^n)'(x)|^\delta \\ &\quad + \frac{h(\alpha^n y)}{h(y)} \big||(\alpha^n)'(x)|^\delta - |(\alpha^n)'(y)|^\delta\big| \\ &\leqq m(h) \left|\frac{1}{h}\right|_\infty A^{\delta + \delta_0} K_{\alpha^n}^{\delta + \delta_0} D(x, y)^{\delta_0} + m\left(\frac{1}{h}\right) |h|_\infty A^\delta K_{\alpha^n}^\delta D(x, y)^{\delta_0} \\ &\quad + |h|_\infty \left|\frac{1}{h}\right|_\infty (1 + \delta) A^\delta K_{\alpha^n}^\delta D(x, y)^{\delta_0}. \end{aligned}$$

The sequence $\left(\dfrac{\|p_{\alpha^n}\|}{K_{\alpha^n}^{\delta}}\right)_{n\in\mathbb{Z}^*}$ is thus bounded. □

Property (R2). Recall that the function f from Λ^0 into $\mathbb{R}$ is defined by $f(x) = B_x(0, a^n 0)$ where a^n is the first term of $\omega(x)$; we extend the definition of f in the following way

$$\forall \alpha \in \mathscr{A},\ \forall n \in \mathbb{Z}^*,\ \forall x \in \Lambda - \Lambda_{\alpha^\pm} \quad f(\alpha^n x) = B_{\alpha^n x}(0, \alpha^n 0)\,.$$

Proposition VIII.4. *One has*

(i) $0 < \nu(f) < +\infty$,

(ii) $\forall n \geqq 1 \quad \sup_{x\in\Lambda} P(f^n)(x) < +\infty\,.$

To prove this proposition we have to estimate functions $f\circ\alpha^n$, $n\in\mathbb{Z}^*$, on each set $\Lambda - \Lambda_{\alpha^\pm}$, $\alpha\in\mathscr{A}$. The following lemma is a direct corollary of lemma VIII.1:

Lemma VIII.5. *There exists $K > 0$ such that for any $\alpha\in\mathscr{A}$ and $n\in\mathbb{Z}^*$ one has*

$$\sup_{x\in\Lambda-\Lambda_{\alpha^\pm}} |f(\alpha^n x)| \leqq K|\mathrm{Log}(K_{\alpha^n})|\,,$$

$$\sup_{\substack{x,y\in\Lambda-\Lambda_{\alpha^\pm}\\ x\neq y}} \frac{|f(\alpha^n x) - f(\alpha^n y)|}{D(x,y)^{\delta_0}} \leqq K\,.$$

Proof of proposition VIII.4. (i) One has

$$\begin{aligned}
\nu(f) &\leqq |h|_\infty \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} \sigma(f 1_{\alpha^n(\Lambda-\Lambda_{\alpha^\pm})})\\
&\leqq |h|_\infty \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} \int_{\Lambda-\Lambda_{\alpha^\pm}} f(\alpha^n x)|(\alpha^n)'(x)|^\delta \sigma(dx)\\
&\leqq K \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} |\mathrm{Log} K_{\alpha^n}| K_{\alpha^n}^\delta \sigma(\Lambda - \Lambda_{\alpha^\pm}) \quad \text{by lemmas VIII.1 and VIII.5}\\
&< +\infty \quad \text{by lemma VIII.1.}
\end{aligned}$$

The fact that $\nu(f) > 0$ is a consequence of the expanding property of T: since there exists $N\in\mathbb{N}^*$ such that $|(T^N)'(x)| \geqq B_0 > 1$ for every $x\in\Lambda^0$ one has $S_N f(x) \geqq \mathrm{Log}\, B_0 > 0$ for every $x\in\Lambda^0$ so that $\nu(S_N f) = N\nu(f) > 0$.

(ii) By lemmas VIII.1 and VIII.5, there exists $C > 0$ such that for any $n\in\mathbb{Z}^*$

$$\|p_{\alpha^n}(f\circ\alpha^n)^l\| \leqq \|p_{\alpha^n}\|\,\|f\circ\alpha^n\|^l \leqq C^l K_{\alpha^n}^\delta |\mathrm{Log}\, K_{\alpha^n}|^l\,.$$

By lemma VIII.1 there exists $\varepsilon > 0$ such that $\sum_{n\geqq 1} K_{\alpha^n}^{\delta-\varepsilon} < +\infty$ for $\alpha\in\mathscr{A}$ which ensures that $\sum_{l\geqq 0}\sum_{n\geqq 1} \dfrac{|t|^l}{l!} |\mathrm{Log}(K_{\alpha^n})|^l K_{\alpha^n}^\delta < +\infty$ as soon as $|t| < \varepsilon$. □

Property (R3). Recall that for any real number $\lambda \in \mathbb{R}$ the operator P_λ is defined by $P_\lambda \varphi = P(e^{i\lambda f}\varphi)$ for every bounded Borel function φ from Λ into $\mathbb{R}$. We show here that P_λ acts on L and we describe the regularity of the map $\lambda \mapsto P_\lambda$.

Proposition VIII.6. *For any $\lambda \in \mathbb{R}$ the operator P_λ acts on L; moreover the mapping $\lambda \mapsto P_\lambda$ is analytic from $\mathbb{R}$ into the space $\left(\mathscr{L}(L), \|\,.\,\|_{\mathscr{L}(L)}\right)$ of continuous linear applications on $(L, \|\,.\,\|)$ with the usual norm.*

Proof. Using lemmas VIII.3 and VIII.5 one obtains

$$\|e^{i\lambda f\circ\alpha^n}\| \leqq |\lambda| m(f\circ\alpha^n) + 1 \leqq K|\lambda| + 1 < +\infty$$

so that $e^{i\lambda f\circ\alpha^n} \in L$; it readily follows that P_λ acts on L. Now, fix $\lambda_0 \in \mathbb{R}$; using lemmas VIII.3 and VIII.5 one can see that for t small enough the series

$$\sum_{l\geqq 0} \frac{|t|^l}{l!} \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} \|p_{\alpha^n}\|\,\|e^{i\lambda_0 f\circ\alpha^n}\|\,\|f\circ\alpha^n\|^l = \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} \|p_{\alpha^n}\|\,\|e^{i\lambda_0 f\circ\alpha^n}\|\,e^{|t|\,\|f\circ\alpha^n\|}$$

converges; consequently one has

$$\left\| P_{\lambda_0+t}(.) - \sum_{l=0}^{N} \frac{(it)^l}{l!} P_{\lambda_0}(f^l.) \right\| \leqq \sum_{l=N+1}^{+\infty} \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} \frac{|t|^l}{l!} \|p_{\alpha^n}\|\,\|e^{i\lambda_0 f\circ\alpha^n}\|\,\|f\circ\alpha^n\|^l \to 0 \quad \text{as} \quad N \to +\infty$$

so that the map $\lambda \mapsto P_\lambda$ is analytic on $\mathbb{R}$. □

Property (R4).

Proposition VIII.7. *One has $P1_\Lambda = 1_\Lambda$, the eigenvalue 1 simple and isolated in the spectrum of P and the corresponding eigenspace is $\mathbb{C}\,1_\Lambda$.*

Proof. Since the probability ν is T-invariant one has $P1_\Lambda(x) = 1_\Lambda(x)$ for ν-*almost all* x in Λ. This property holds in fact for *every* point in Λ because P acts on L.

The description of the spectrum of P on L is based on the following theorem, due to Ionescu-Tulcea and Marinescu and whose modern formulation can be found in [18].

Theorem (Ionescu-Tulcea and Marinescu)**.** *Let $(E, \|.\|_E)$ be a $\mathbb{C}$-Banach space and Q a linear continuous operator on $(E, \|.\|_E)$ whose spectral radius is $\leqq 1$. Assume that there exists on E a norm $|.|$ such that*

(i) *the operator Q is compact from $(E, \|.\|_E)$ into $(E, |.|)$,*

(ii) *there exist $0 < r < 1$, $R > 0$ and $N \in \mathbb{N}^*$ such that Q satisfies the following inequality*:

$$\forall \varphi \in E \quad \|Q^N\varphi\|_E \leqq r\|\varphi\|_E + R|\varphi|.$$

Then Q admits at most a finite number of modulus one eigenvalues, the associated eigenspaces are finite dimensional and the rest of the spectrum of Q on $(E, \|.\|_E)$ is included in a disc of radius strictly less than 1.

In order to control the spectrum of P on L we need the following

Lemma VIII.8. *There exist $0<r<1$, $R>0$ et $N \in \mathbb{N}^*$ such that*

$$\forall \varphi \in L \quad \|P^N \varphi\| \leqq r\|\varphi\| + R|\varphi|_\infty .$$

Iterating this inequality, one proves that $(\|P^n\|)_{n \geqq 1}$ is bounded so that $\lim_{n \to +\infty} \|P^n\|^{1/n} \leqq 1$; consequently, by Ionescu-Tulcea and Marinescu's theorem, the operator P on L has at most a finite number of modulus one eigenvalues, the corresponding eigenspaces are finite dimensional and the rest of the spectrum is included in a disc of radius strictly less than 1. Proposition VIII.7 follows, thanks to the

Lemma VIII.9. *Let $\varphi \in L$ such that $P\varphi = e^{i\theta}\varphi$. If $\# \mathscr{A} \geqq 3$ one thus has $e^{i\theta} = 1$ and $\varphi = C1_\Lambda$, $C \in \mathbb{C}$; otherwise, $\# \mathscr{A} \geqq 2$ and there are two cases*:

– $e^{i\theta} = 1$ *and* $\varphi \in \mathbb{C}\, 1_\Lambda$,

– $e^{i\theta} = -1$ *and* $\varphi \in \mathbb{C}\, (1_{\Lambda_{\alpha_1^\pm}} - 1_{\Lambda_{\alpha_2^\pm}})$.

It remains to establish lemmas VIII.8 and 9.

Proof of lemma VIII.8. By proposition V.1 there exists $N \geqq 1$ such that $\inf_{x \in \Lambda^0} |(T^N)'(x)| = B_0 > 1$. For $\alpha \in \mathscr{A}$ set

$$\mathscr{A}_N(\alpha) = \{\bar{a} = (a_1, \ldots, a_N) \in \mathscr{A}^N \,|\, a_{j+1} \neq a_j \text{ for } 1 \leqq j < N \text{ and } a_N \neq \alpha\};$$

for $x, y \in \Lambda_\alpha$, $\bar{a} = (a_1, \ldots, a_N) \in \mathscr{A}_N(\alpha)$ and $\bar{k} = (k_1, \ldots, k_N) \in (\mathbb{Z}^*)^N$ we thus have

$$D\,(a_1^{k_1} \cdots a_N^{k_N} x, a_1^{k_1} \cdots a_N^{k_N} y) \leqq \frac{1}{B_0}\, D(x,y)\,.$$

Set $p_{\bar{a}\bar{k}}(x) = p_{a_1^{k_1}}(a_2^{k_2} \cdots a_N^{k_N} x)\, p_{a_2^{k_2}}(a_3^{k_3} \cdots a_N^{k_N} x) \cdots p_{a_N^{k_N}}(x)$; for $\varphi \in L$ and $x, y \in \Lambda_\alpha$ we have

$$\begin{aligned}
|P^N\varphi(x) - P^N\varphi(y)| &\leqq \sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} p_{\bar{a}\bar{k}}(x)\,|\varphi(a_1^{k_1} \cdots a_N^{k_N} x) - \varphi(a_1^{k_1} \cdots a_N^{k_N} y)| \\
&\quad + |\varphi|_\infty \sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} |p_{\bar{a}\bar{k}}(x) - p_{\bar{a}\bar{k}}(y)| \\
&\leqq m(\varphi) \sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} p_{\bar{a}\bar{k}}(x)\, D(a_1^{k_1} \cdots a_N^{k_N} x, a_1^{k_1} \cdots a_N^{k_N} y)^{\delta_0} \\
&\quad + |\varphi|_\infty \sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} |p_{\bar{a}\bar{k}}(x) - p_{\bar{a}\bar{k}}(y)| \\
&\leqq \left(\frac{1}{B^{\delta_0}}\, m(\varphi) + |\varphi|_\infty \sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} \frac{|p_{\bar{a}\bar{k}}(x) - p_{\bar{a}\bar{k}}(y)|}{D(x,y)^{\delta_0}} \right) D(x,y)^{\delta_0}.
\end{aligned}$$

Note that $\sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} |p_{\bar{a}\bar{k}}(x) - p_{\bar{a}\bar{k}}(y)| \leqq \sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} \sum_{s=1}^{N} M_{\bar{a}\bar{k}}(s, x, y)$ with

$$\begin{aligned} M_{\bar{a}\bar{k}}(s, x, y) = {} & p_{a_1^{k_1} \dots a_{s-1}^{k_{s-1}}}(a_s^{k_s} \cdots a_N^{k_N} x) \\ & \times |p_{a_s^{k_s}}(a_{s+1}^{k_{s+1}} \cdots a_N^{k_N} x) - p_{a_s^{k_s}}(a_{s+1}^{k_{s+1}} \cdots a_N^{k_N} y)| \, p_{a_{s+1}^{k_{s+1}} \dots a_N^{k_N}}(y) \\ \leqq {} & p_{a_1^{k_1} \dots a_{s-1}^{k_{s-1}}}(a_s^{k_s} \cdots a_N^{k_N} x) \, m(p_{a_s^{k_s}}) \, C^{N-r} D(x, y)^{\delta_0} p_{a_{s+1}^{k_{s+1}} \dots a_N^{k_N}}(y) \end{aligned}$$

where $C \in [1, +\infty[$ is such that $D(a^n x, a^n y)^{\delta_0} \leqq C D(x, y)^{\delta_0}$ for every $a \in \mathscr{A}$, $x, y \in \Lambda - \Lambda_{a^\pm}$ and $n \in \mathbb{Z}^*$; one thus obtains

$$m(P^N \varphi) \leqq \frac{1}{B_0^{\delta_0}} m(\varphi) + \frac{C^{N+1}}{C-1} \sum_{\alpha \in \mathscr{A}} \sum_{n \in \mathbb{Z}^*} \|p_{\alpha^n}\| |\varphi|_\infty$$

and lemma VIII.8 follows with $r = \dfrac{1}{B_0^{\delta_0}}$ and $R = 1 + \dfrac{C^{N+1}}{C-1} \sum_{\alpha \in \mathscr{A}} \sum_{n \in \mathbb{Z}^*} \|p_{\alpha^n}\|$. □

Proof of lemma VIII.9. Let $\varphi \in L$ and $\theta \in \mathbb{R}$ such that $P\varphi = e^{i\theta}\varphi$. Equalities $P\varphi = e^{i\theta}\varphi$ and $\nu P = \nu$ imply $P|\varphi|(x) = |\varphi(x)| \, \nu(dx) \, p.s$; since φ and $P\varphi$ belong to L and the support of ν is Λ, this last equality holds for any x in Λ.

Suppose that $\# \mathscr{A} \geqq 3$. Let y, y' in Λ such that $|\varphi(y)| = \sup_{x \in \Lambda} |\varphi(x)|$, $|\varphi(y')| = \inf_{x \in \Lambda} |\varphi(x)|$ and consider $\alpha \in \mathscr{A}$ such that y and y' do not belong to $\Lambda_{\alpha^\pm}$ (such an α does exist since $\# \mathscr{A} \geqq 3$); by a convexity argument we have $|\varphi(y)| = |\varphi(\alpha^n y)|$ and $|\varphi(y')| = |\varphi(\alpha^n y')|$ for every $n \in \mathbb{Z}^*$. Letting $n \to +\infty$, one obtains $|\varphi(y)| = |\varphi(y')| = |\varphi(x_\alpha)|$ which ensures that $|\varphi|$ is constant on Λ. Assume $|\varphi| \neq 0$; for every $\alpha \in \mathscr{A}$ and $x \in \Lambda_{\alpha^\pm}$ one has $e^{i\theta} = \sum_{\beta \in \mathscr{A}} \sum_{n \in \mathbb{Z}^*} p_{\beta^n}(x) \dfrac{\varphi(\beta^n x)}{\varphi(x)}$ and so $\varphi(x) = e^{-i\theta}\varphi(\beta^n x)$ for every $\beta \neq \alpha$ and $n \in \mathbb{Z}^*$; letting $n \to +\infty$ one obtains $\varphi(x) = e^{-i\theta}\varphi(x_\beta)$ which proves that φ is constant on Λ and that $e^{i\theta} = 1$.

Suppose now $\mathscr{A} = \{\alpha_1, \alpha_2\}$. We have $P(1_{\Lambda_{\alpha_1^\pm}}) = 1_{\Lambda_{\alpha_2^\pm}}$ and $P(1_{\Lambda_{\alpha_2^\pm}}) = 1_{\Lambda_{\alpha_1^\pm}}$. Moreover

$$\forall x \in \Lambda_{\alpha_1^\pm} \quad P^2\varphi(x) = \sum_{n, m \in \mathbb{Z}^*} p_{\alpha_2^n}(x) \, p_{\alpha_1^m}(\alpha_2^n x) \, \varphi(\alpha_1^m \alpha_2^n x) = e^{2i\theta}\varphi(x).$$

Let y_1 and y_1' in $\Lambda_{\alpha_1^\pm}$ such that $|\varphi(y_1)| = \sup_{y \in \Lambda_{\alpha_1^\pm}} |\varphi(y)|$ and $|\varphi(y_1')| = \inf_{y \in \Lambda_{\alpha_1^\pm}} |\varphi(y)|$; by a convexity argument we have $|\varphi(y_1)| = |\varphi(\alpha_1^m \alpha_2 y_1)|$ and $|\varphi(y_1')| = |\varphi(\alpha_1^m \alpha_2 y_1')|$ for every $m \in \mathbb{Z}^*$. Letting $m \to +\infty$, one obtains that $|\varphi|$ is constant on $\Lambda_{\alpha_1^\pm}$. If $|\varphi| \neq 0$ on $\Lambda_{\alpha_1^\pm}$, one has $\varphi(x) = e^{-2i\theta}\varphi(x_{\alpha_1})$ which proves that φ is constant on $\Lambda_{\alpha_1^\pm}$ and $e^{2i\theta} = 1$. The same conclusion holds on $\Lambda_{\alpha_2^\pm}$ which finishes the proof. □

Property (R5). We describe here the top of the spectrum on L of the operators P_λ, $\lambda \neq 0$.

Proposition VIII.10. *For any $\lambda \in \mathbb{R}^*$, the spectral radius of P_λ on $(L, \|.\|)$ is strictly less than* 1.

Proof. Let $\varphi \in L$; we have $\forall x \in \Lambda \quad P\varphi(x) = \sum_{\alpha \in \mathscr{A}} \sum_{n \in \mathbb{Z}^*} p_{\alpha^n}(x)\, e^{i\lambda f(\alpha^n x)} \varphi(\alpha^n x)$. To control the spectrum of P_λ we prove that P_λ satisfies hypotheses of Ionescu-Tulcea and Marinescu's theorem; if one replaces p_{α^n} with $p_{\alpha^n} e^{i\lambda f \circ \alpha^n}$ in the proof of lemma VIII.8 one shows that there exist $r \in]0,1[$ and $A, B > 0$ such that

$$\forall \varphi \in L \quad \|P_\lambda^N \varphi\| \leqq r\|\varphi\| + (A|\lambda| + B)|\varphi|_\infty .$$

Iterating this inequality, one obtains that $(P_\lambda^n)_{n \geqq 1}$ is bounded in $(L, \|.\|)$; the spectral radius of P_λ is thus $\leqq 1$ and proposition VIII.10 is a consequence of the following

Lemma VIII.11. *For $\lambda \neq 0$ the operator P_λ does not admit eigenvalues of modulus one.*

Proof. The equality $P_\lambda \varphi = e^{i\theta}\varphi$ gives $P|\varphi| = |\varphi|$ and so, by lemma VIII.9, $|\varphi|$ is constant on Λ. By a convexity argument $e^{i\theta}\varphi(x) = e^{i\lambda f(\alpha^n x)}\varphi(\alpha^n x)$ so that $\lim_{n \to +\infty} e^{i\lambda f(\alpha^n x)}$ does exist for every $\alpha \in \mathscr{A}$ and $x \in \Lambda - \Lambda_{\alpha^\pm}$. Suppose that α is parabolic; one has $\lim_{n \to +\infty} f(\alpha^n x) = +\infty$ and $\lim_{n \to +\infty} f(\alpha^{n+1} x) - f(\alpha^n x) = 0$. Consequently, for any $a > 0$ there exists a sequence $(n_k)_{k \geqq 1}$ of integers such that $\lim_{k \to +\infty} f(\alpha^{n_{k+1}} x) - f(\alpha^{n_k} x) = a$; it follows $e^{i\lambda a} = 1$; and so $\lambda = 0$. (Note that the existence of parabolic transformations in Γ is essential in the present proof.) □

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