Mathematische Zeitschrift
https://doi.org/10.1007/s00209-018-2199-6

Mathematische Zeitschrift

Entropy rigidity of negatively curved manifolds of finite volume

M. Peigné[1] · **A. Sambusetti**[2]

Received: 6 February 2018 / Accepted: 7 November 2018

Abstract
We prove the following entropy-rigidity result in finite volume: if X is a negatively curved manifold with curvature $-b^2 \leq K_X \leq -1$, then $Ent_{top}(X) = n - 1$ if and only if X is hyperbolic. In particular, if X has the same length spectrum of a hyperbolic manifold X_0, the it is isometric to X_0 (we also give a direct, entropy-free proof of this fact). We compare with the classical theorems holding in the compact case, pointing out the main difficulties to extend them to finite volume manifolds.

Keywords Negative curvature · Entropy · Length spectrum · Bowen–Margulis measure

Mathematics Subject Classification: 53C20 · 37C35

1 Introduction

The problem of length spectrum rigidity of Riemannian manifolds has a long history. The fact that, in negative curvature (even in constant curvature), the collection of the lengths of all closed geodesics, together with all multiplicites, does not determine the metric is well known since [37]. On the other hand, on a *compact*, negatively curved surface $\bar{X}$, the metric is determined up to isometry by the marked length spectrum (that is, the map $\mathcal{L} : \mathcal{C}(\bar{X}) \to \mathbb{R}$ associating to each free homotopy class of loops in $\bar{X}$ the length of the shortest geodesic in the class); this was proved by Otal [27] and, independently, by Croke [11]. The same is true in dimension $n \geq 3$ for any compact, *locally symmetric* manifold $\bar{X}_0$ of negative curvature: the locally symmetric metric on $\bar{X}_0$ is determined, among all negatively curved metrics, by its marked length spectrum. This is consequence of Besson–Courtois–Gallot's solution of the minimal entropy conjecture and of the fact, proved by Hamenstadt [20], that if a compact,

✉ A. Sambusetti
sambuset@mat.uniroma1.it

M. Peigné
peigne@univ-tours.fr

1 LMPT, UMR 6083, Faculté des Sciences et Techniques, Parc de Grandmont, 37200 Tours, France

2 Istituto di Matematica G. Castelnuovo, Sapienza Università di Roma, P.le Aldo Moro 5, 00185 Rome, Italy

negatively curved manifold $\bar{X}$ has the same marked length spectrum as a compact, locally symmetric space $\bar{X}_0$, then $\mathrm{vol}(\bar{X}) = \mathrm{vol}(\bar{X}_0)$.[1]

Less seems known about the length rigidity of negatively curved, *finite-volume* manifolds: most generalizations are not straightforward, and seem to require additional assumptions (such as bounds on the curvature and on its derivatives, or the finiteness of the Bowen–Margulis measure); we will try to point out some of these difficulties throughout the paper. For instance, the fact that having the same marked length spectrum implies the existence of a C^0-conjugacy of the geodesic flow, would certainly require some new arguments for finite-volume manifolds.[2]

Theorem 1.1 *Let $\bar{X}$ be a finite volume n-manifold with pinched, negative curvature $-b^2 \leq K_{\bar{X}} \leq -1$ which is homotopically equivalent to a locally symmetric manifold $\bar{X}_0$, with curvature normalized between -4 and -1. If $\bar{X}$ and $\bar{X}_0$ have same marked length spectrum, then they are isometric.*

The proof of this is probably known to experts and follows a classical scheme: one can construct a Γ-equivariant map $f : X \to X_0$ between the universal coverings, which induces a homeomorphism between the boundaries and preserves the cross-ratio; then, the conclusion stems, for instance, from Bourdon's result [6] on Möbius embeddings from locally symmetric to $CAT(-1)$-spaces. However, the main difficulty, in the case of finite volume manifolds, is to show that f is a quasi-isometry, the quotients $\bar{X}$ and $\bar{X}_0$ being non-compact; we will give a short proof of this fact in Sect. 3, by way of example, to measure the difference from the compact case.

It is tempting to approach the above problem by using a finite-volume version of Besson-Courtois-Gallot's entropy inequality, given by Storm [35]. For this, recall that the *volume entropy* of $\bar{X}$ is the exponential growth rate of the volume of balls in X, defined as

$$Ent(\bar{X}) = \limsup_{R \to +\infty} \frac{1}{R} \ln vol(B_X(\mathbf{x}, R))$$

where $B_X(\mathbf{x}, R)$ denotes the open ball of X with radius R, centered at the point $\mathbf{x}$. The volume entropy $Ent(\bar{X})$ is related to the *topological entropy* $Ent_{top}(\bar{X})$ of the geodesic flow Φ_g on $U\bar{X}$. The notion of topological entropy of a flow Φ_g on a metrizable topological space $\bar{Y}$ is well known when the space is compact, and it has been extended in [8] to the case when it is non compact. Following R. Bowen, one first defines the entropy $Ent(\Phi_g, d)$ of the flow Φ_g, relatively to some distance d compatible with the topology of $\bar{Y}$; this number depends on d and is called the *entropy relative to* d; then, the *topological entropy* $Ent_{top}(\Phi_g)$ is defined as the infimum of the entropies with respect to all compatible distances d on $\bar{Y}$. It is well known that, in negative curvature, the topological entropy of the restriction of the geodesic flow to its non wandering set equals the *critical exponent* of the group $\Gamma = \pi_1(X)$ acting on the universal covering:

$$\delta_\Gamma := \lim_{R \to \infty} \frac{1}{R} \ln \#\{\gamma \in \Gamma | d(x, \gamma x) \leq R\}$$

[1] By [18], two compact, negatively curved manifolds having the same marked length spectrum have C^0-conjugated geodesic flow; moreover, if a compact manifold $\bar{X}$ has geodesic flow which is C^0-conjugated to the flow of a manifold $\bar{X}_0$ whose unitary tangent bundle has a C^1-Anosov splitting (e.g., a locally symmetric space), then $\bar{X}$ has the same volume as $\bar{X}_0$, see [20]. The fact that, for compact manifolds, volume is preserved under C^1-conjugacies is much easier and relies on Stokes' formula, cp. [12].

[2] Cp. Lemma 2.4 in [18], which is central in the argument: it is based on the fact that the closed geodesics on $\bar{X}$ equidistribute towards the Bowen–Margulis measure when $\bar{X}$ is compact. This property does not hold for non uniform lattices as soon as the Bowen–Margulis measure is infinite.

and complies with a variational principal, as proved in [29]. Recall that the non-wandering set of the geodesic flow of negatively curved finite-volume manifolds equals all $U\bar{X}$ so $Ent_{top}(\bar{X}) = Ent_{top}(\Phi_g) = \delta_\Gamma$, and this also equals the exponential growth of the function counting the number of closed geodesics of $\bar{X}$ with length $\leq R$. As a consequence, the inequality $Ent(\bar{X}) \geq Ent_{top}(\bar{X}) = \delta_\Gamma$ always holds. It is classic that for compact, negatively curved manifolds, one always has the equality $Ent(\bar{X}) = Ent_{top}(\bar{X})$, see [26]; however, when $\bar{X}$ has finite volume but is non-compact, $Ent(\bar{X})$ can be strictly greater than $Ent_{top}(\bar{X})$, as shown in [14], [15].

In [35] Storm, generalizing [1], proves the inequality

$$Ent(\bar{X})^n Vol(\bar{X}) \geq Ent(\bar{X}_0)^n Vol(\bar{X}_0),$$

for finite-volume manifolds X admitting a proper map of non-zero degree on a finite volume, locally symmetric manifold $\bar{X}_0$ of negative curvature, and showing that the equality holds only if $\bar{X}$ is isometric to $\bar{X}_0$. This result concerns the volume entropies[3] of $\bar{X}$, and not the topological entropy. Therefore, one cannot apply directly this result to deduce Theorem 1.1, since the volume entropy $Ent(\bar{X})$ (unlike the topological entropy) is not a-priori preserved by the condition of having same marked length spectrum, or by a conjugacy of the geodesic flows. Moreover, it is not clear whether, for finite volume manifolds, the volume is preserved under a conjugacy of the flows.[4]

The upper bound on the curvature $K_{\bar{X}} \leq -1$ in Theorem 1.1 seems unreasonably strong, as it implies, when $\bar{X}_0$ is hyperbolic, that $\bar{X}$ has marked length spectrum which is *asymptotically critical*: that is, its exponential growth rate δ_Γ is greater than or equal to the corresponding exponential growth rate for $\bar{X}_0$ (cp. [14], Lemma 4.1). We expect that the same result holds without curvature bounds, but, even in this weaker form, we were unable to find a proof of this result in literature.

The knowledge of the full marked length spectrum can be relaxed, as we show in the following result (which implies Theorem 1.1 in the real hyperbolic case):

Theorem 1.2 *Let $\bar{X}$ be a finite volume n-manifold with pinched, negative curvature $-b^2 \leq K_{\bar{X}} \leq -1$. Then $Ent_{top}(\bar{X}) \geq n-1$, and the equality $Ent_{top}(\bar{X}) = n-1$ holds if and only if $\bar{X}$ is hyperbolic.*

The entropy characterization of constant curvature (and locally symmetric) metrics has been declined in many different ways so far: in the compact case, the above theorem is due to Knieper (see [24], where this result is not explicitly stated, but can be established following the argument of the proof of Theorem 5.2); see also [4,10] for a proof in the convex-cocompact case.

We want to stress here that a basic difference between Theorem 1.2 (or, more precisely, their compact versions in [4,10,24]) and the celebrated entropy characterization of Hamenstädt [19] of locally symmetric metrics, with same curvature normalization, is the lack of any locally symmetric manifold $\bar{X}_0$ of reference homotopically equivalent to $\bar{X}$. Actually, the characterization given by Theorem 1.2 is very particular to constant curvature spaces and it does not generalize, as it is, to locally symmetric spaces: indeed, it is easy to construct compact, pinched, negatively curved manifolds with $-b^2 \leq K_X \leq -1$ having same entropy as, let's say, the complex hyperbolic space, but which are not complex hyperbolic.

3 Cp. the definition of the maps Ψ_c^b in [35], which clearly require that c is greater than the exponential growth rate of the universal covering of the manifold under consideration.

4 This seems unclear even under the assumption of a C^1-conjugacy; cp. the proof of Proposition 1.2 in [12], where Stokes's theorem fails, unless one knows that the conjugacy F has bounded derivatives.

The same difference holds with the existing, finite volume versions of Besson-Courtois-Gallot's characterization of locally symmetric spaces, in particular with Boland-Connell-Souto's papers [3] and Storm's [35]: these two works, together, imply that if a finite volume manifold $\bar{X}$ with curvature $K_{\bar{X}} \leq -1$ has volume entropy $Ent(\bar{X}) = n-1$, then it is hyperbolic, provided that one knows beforehand that $\bar{X}$ is homotopically equivalent to a hyperbolic manifold $\bar{X}_0$. Besides the difference between volume and topological entropy stressed above, this strong supplementary topological assumption on $\bar{X}$ is not made in Theorem 1.2.

Let us also point out that Knieper's approach in [24] does not allow to deduce the above characterization in the finite volume case. Although G. Knieper's horospherical measure μ_H can be perfectly defined in this context (following section of [24]), it can easily be infinite, as well as the Bowen–Margulis measure μ_{BM}: given a finite volume surface $\bar{X}$ with convergent fundamental group Γ and with a cusp whose metric, in horospherical coordinates, writes as $\mathcal{A}^2(t)dx^2 + dt^2$, it is not difficult to show that μ_H is infinite as soon as

$$\int_0^\infty e^{\delta_\Gamma t} \mathcal{A}(t)dt = +\infty$$

(cp. examples in Section 3, [15]). Therefore, all formulas in [24] relating $Ent_{top}(\bar{X})$ to the trace of the second fundamental form of unstable horospheres need to be justified in some other way.[5]

On the other hand, we will give in Sect. 4 a proof of Theorem 1.2 using the barycenter method, initiated by Besson–Courtois–Gallot in [1,2], together with some careful estimates of the Patterson-Sullivan measure, which will not need neither the finiteness of μ_{BM} (or μ_H) nor the conservativity of the geodesic flow with respect to μ_{BM}.

Also, notice that if we drop the assumption $K_{\bar{X}} \geq -b^2$ in Theorem 1.2, the manifold $\bar{X}$ might as well be of infinite type (i.e. with infinitely generated fundamental group, or even without any cusp, see examples in [30]), hence very far from being a hyperbolic manifold of finite-volume.

Notations. Given functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, *we will systematically write* $f \overset{C}{\prec} g$ *(or* $g \overset{C}{\succ} f$*) if there exists* $C > 0$ *and* $R_0 > 0$ *such that* $f(R) \leq Cg(R)$ *for* $R > R_0$. *We write* $f \overset{C}{\asymp} g$ *when* $g \overset{C}{\prec} f \overset{C}{\prec} g$ *for* $R \gg 0$ *(or simply* $f \asymp g$ and $f \prec g$ *when the constants* C *and* R_0 *are unessential)*

2 Geometry at infinity in negative curvature

Throughout all the paper, X will be a n-dimensional, complete, simply connected manifold with strictly negative curvature $-b^2 \leq K_X \leq -1$.

Let $X(\infty)$ the ideal boundary of X: for $x, y \in X$ and $\xi \in X(\infty)$, we will denote by $[x, y]$ (resp. $[x, \xi[$) the geodesic segment from x to y (resp. the ray from x to ξ), and by $x\xi(t)$ the parametrization of geodesic ray $[x, \xi[$ by arc length. Let

$$b_\xi(x, y) = \lim_{z\to\xi} d(x, z) - d(z, y)$$

be the Busemann function centered at ξ; the level set $\partial H_\xi(x) = \{y \mid b_\xi(x, y)| = 0\}$ (resp. the suplevel set $H_\xi(x) = \{y \mid b_\xi(x, y) \geq 0\}$ is the horosphere (resp. the horoball) with center ξ

[5] For instance, Corollary 4.2 in [24] only holds for μ_H-integrable functions, and cannot be applied as it is to constant functions or to $tr\, U^+(v)$ to deduce Theorem 5.1, when $\|\mu_H\| = \infty$.

and passing through x. We will denote by d_ξ the horospherical distance between two points on a same horosphere centered at ξ, and we define the radial semi-flow $(\psi_{\xi,t})_{t\geq 0}$ in the direction of ξ as follows: for any $x \in X$, the point $\psi_{\xi,t}(x)$ lies on the geodesic ray $[x, \xi[$ at distance t from x.

Finally, recall that for any fixed $x \in X$, the Gromov product between two points $\xi, \eta \in X(\infty)$, $\xi \neq \eta$, is defined as

$$(\xi|\eta)_x = \frac{b_\xi(x, y) + b_\eta(x, y)}{2}$$

where y is any point on the geodesic $]\xi, \eta[$ joining ξ to η; as $K_X \leq -1$, the expression $D_x(\xi, \eta) = e^{-(\xi|\eta)_x}$ defines a distance on $X(\infty)$, which we will call the visual distance from x, cp. [5]. Accordingly, the *cross-ratio* on $X(\infty)^4$ is defined as

$$[\xi_1, \xi_2, \xi_3, \xi_4] = \frac{D_x(\xi_1, \xi_3)D_x(\xi_2, \xi_4)}{D_x(\xi_1, \xi_4)D_x(\xi_2, \xi_3)} = \lim_{\substack{p_1,p_2,p_3,p_4\in X \\ (p_1,p_2,p_3,p_4)\to(\xi_1,\xi_2,\xi_3,\xi_4)}} e^{d(p_1,p_3)+d(p_2,p_4)-d(p_1,p_4)-d(p_2,p_3)}$$

for all $\xi_1, \xi_2, \xi_3, \xi_4 \in X(\infty)$, and it is easily seen that it is independent from the choice of the base point x, cp. [28], [5].

We will repeatedly make use of the following, classical result in strictly negative curvature: there exists $\epsilon(\vartheta) = \log(\frac{2}{1-\cos\vartheta})$ such that any geodesic triangle xyz in X making angle $\vartheta = \angle_z(x, y)$ at z satisfies:

$$d(x, y) \geq d(x, z) + d(z, x) - \epsilon(\vartheta). \tag{1}$$

2.1 On the geometry of finite volume manifolds

Consider a lattice Γ of X. The quotient manifold $\bar{X} = \Gamma\backslash X$ has finite volume, it is thus a geometrically finite manifold which admits some particular decomposition which we now recall. The following classical results are due to Bowditch [7], and we state them in the particular case of finite volume manifolds:

a. The limit set of $L(\Gamma)$ of Γ is the full boundary at infinity $X(\infty)$ and is the disjoint union of the radial limit set $L_{rad}(\Gamma)$ with finitely many orbits of *bounded* parabolic fixed points $L_{bp}(\Gamma) = \Gamma\xi_1 \cup \ldots \cup \Gamma\xi_l$; this means that each $\xi_i \in L_{bp}(\Gamma)$ is the fixed point of some maximal parabolic subgroup P_i of Γ, acting co-compactly on $X(\infty) \setminus \{\xi_i\}$;
b. *(Margulis' lemma)* there exist closed horoballs $H_{\xi_1}, \ldots, H_{\xi_l}$ centered respectively at $\xi_1, \ldots, \xi_l$, such that $\gamma H_{\xi_i} \cap H_{\xi_j} = \emptyset$ for all $1 \leq i, j \leq l$ and all $\gamma \in \Gamma \setminus P_i$;
c. The finite volume manifold $\bar{X}$ can be decomposed into a disjoint union of a compact set $\bar{\mathcal{K}}$ and finitely many "cusps" $\bar{\mathcal{C}}_1, \ldots, \bar{\mathcal{C}}_l$: each $\bar{\mathcal{C}}_i$ is isometric to the quotient of H_{ξ_i} by a corresponding maximal bounded parabolic group P_i. We refer to $\bar{\mathcal{K}}$ and to $\bar{\mathcal{C}} = \cup_i \bar{\mathcal{C}}_i$ as to the *thick part* and the *cuspidal part* of $\bar{X}$.

For any fixed $x \in X$, let $\mathcal{D} = \mathcal{D}(\Gamma, x)$ the Dirichlet domain of Γ centered at x; this is a convex fundamental subset of X, and we may assume that $\mathcal{D}$ contains the geodesic rays $[x, \xi_i[$. Each parabolic group P_i acts co-compactly on the horosphere ∂H_{ξ_i} which bounds the horoball H_{ξ_i}; setting $\mathcal{S}_i = \mathcal{D} \cap \partial H_{\xi_i}$ and $\mathcal{C}_i = \mathcal{D} \cap H_{\xi_i} \simeq \mathcal{S}_i \times \mathbb{R}_+$, the fundamental domain $\mathcal{D}$ can be decomposed into a disjoint union

$$\mathcal{D} = \mathcal{K} \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_l \tag{2}$$

where $\mathcal{K}$ is a convex, relatively compact set containing x in its interior and projecting to the thick part $\bar{\mathcal{K}}$ of $\bar{X}$, while $\mathcal{C}_i$ and $\mathcal{S}_i$ are, respectively, connected fundamental domains for the action of P_i on H_{ξ_i} and ∂H_{ξ_i}, projecting respectively to $\bar{\mathcal{C}}_i$ and $\bar{\mathcal{S}}_i$.

2.2 Growth of parabolic subgroups

The subgroups P_1, s, P_l will play a crucial role in the sequel; the growth of their orbital functions is best expressed by introducing the horospherical area function. Let us recall the necessary definitions:

Definition 2.1 (*Horospherical Area*) Let P be a *bounded* parabolic group of isometries of X fixing $\xi \in X(\infty)$: that is, P acts cocompactly on $X(\infty) \setminus \{\xi\}$ (as well as on every horosphere centered at ξ). Given $x \in X$, let $\mathcal{S}_x$ be a fundamental, relatively compact domain for the action of P on $\partial H_\xi(x)$: the *horospherical area function* of P is the function

$$\mathcal{A}_P(x, R) = \mathrm{vol}\left[P\backslash\psi_{\xi,R}\left(\partial H_\xi(x)\right)\right] = \mathrm{vol}\left[\psi_{\xi,R}\left(\mathcal{S}_x\right)\right]$$

where vol denotes the Riemannian measure of horospheres.

Remark 2.2 When $-b^2 \le K_X \le -a^2 < 0$, well-known estimates of the differential of the radial flow (cp. [21]) yield, for any $t \in \mathbb{R}$ and $v \in T^1X$

$$e^{-bt} \parallel v \parallel \le \parallel d\psi_{\xi,t}(v) \parallel \le e^{-at} \parallel v \parallel \tag{3}$$

Therefore we deduce that, for any $\Delta > 0$,

$$e^{-(n-1)b\Delta} \le \frac{\mathcal{A}_P(x, R+\Delta)}{\mathcal{A}_P(x, R)} \le e^{-(n-1)a\Delta} \tag{4}$$

The following Proposition shows how the horospherical area $\mathcal{A}_P$ is related to the orbital function of P, cp. [14]:

Proposition 2.3 *Let P be a bounded parabolic group of X fixing ξ, with $diam(\mathcal{S}_x) \le d$. There exist R_0 and Δ_0 only depending on n, a, b, d and constants $C = C(n, a, b, d)$ and $C' = C'(n, a, b, d, \Delta)$ such that, for any $R \ge b_\xi(x, y) + R_0$ and any $\Delta > \Delta_0$, the numbers $v_P(x, y, R)$ and $v_P^\Delta(x, y, R)$ of orbit points of Py falling, respectively, in the balls $B(x, R)$ and in the annuli $A^\Delta(x, R)$ satisfy:*

$$v_P(x, y, R) = \{p \in P \mid d(x, py) < R\} \overset{C}{\asymp} \mathcal{A}_P^{-1}\left(x, \frac{R + b_\xi(x, y)}{2}\right)$$

$$v_P^\Delta(x, y, R) = \left\{p \in P \mid R - \frac{\Delta}{2} \le d(x, py) \le R + \frac{\Delta}{2}\right\} \overset{C'}{\asymp} \mathcal{A}_P^{-1}\left(x, \frac{R + b_\xi(x, y)}{2}\right).$$

3 Length spectrum and rigidity

This section is devoted to the proof of Theorem 1.1.

Let Γ be the fundamental group of the manifolds $\bar{X}$ and $\bar{X}_0$, acting by isometries on their Riemannian universal coverings X and X_0 respectively. We will construct a Γ-equivariant homeomorphism $f_\infty : \partial X(\infty) \to X_0(\infty)$ and apply the following:

Theorem 3.1 *(cp. [6]) Let X be a* CAT(-1)*-space and X_0 a symmetric space of rank one, with curvature $-4 \leq K_{X_0} \leq 1$. Assume that $f_\infty : X(\infty) \to X_0(\infty)$ is a Γ-equivariant homeomorphism which preserves the cross-ratio: then there exists a Γ-equivariant isometry $f : X \to X_0$ whose extension on $X(\infty)$ coincides with f_∞.*

For this, we fix $x \in X$ and $x_0 \in X_0$ and consider the natural Γ-equivariant bijection $\phi : \Gamma x \to \Gamma x_0$. The main difficulty here is to show the following:

Proposition 3.2 *The map ϕ is a quasi-isometry between the orbits, with respect to the distances induced by the Riemannian distances of X and X_0 respectively.*

We assume Proposition 3.2 for a moment. Since $\bar{X}$ and $\bar{X}_0$ have finite volume, the limit set of Γ coincides with the full boundaries $X(\infty)$ and $X_0(\infty) = \mathbb{S}^{n-1}$ and the map ϕ extends to a bi-Hölder and Γ-equivariant homeomorphism f_∞ between these boundaries, endowed with their natural visual metric from x and x_0.

Now, the fact that $\bar{X}$ and $\bar{X}_0$ have the same marked length spectrum implies that f_∞ preserves the cross ratio; this follows for instance from [28]. For the sake of completeness, we will give a proof of this fact at the end of this section (Proposition 3.5), based on an argument from [23] (where the same is proved for symmetric spaces).

We conclude by 3.1 that there exists an isometry between the quotients $\bar{X}$ and $\bar{X}_0$. $\square$

Proof of Proposition 3.2 Let us first show that there exists $\lambda > 1$ such that, for all $\gamma \in \Gamma$, we have

$$d_0(x_0, \gamma x_0) \leq \lambda d(x, \gamma x) + \lambda. \tag{5}$$

Consider the decomposition of $\bar{X}$ described in Sect. 2.1: we denote by $\mathcal{H}$ the set of pairwise disjoint horoballs which project on the cuspidal part of $\bar{X}$, so that $\widetilde{\mathcal{K}} := X \setminus \cup_{H \in \mathcal{H}} H = \Gamma\mathcal{K}$ is the subset of X projecting to the thick part $\bar{\mathcal{K}}$ of $\bar{X}$.

We assume that $d(H, H') \geq 1$ for any $H \neq H'$ in $\mathcal{H}$, and set diam$(\mathcal{K}) = D$.

For any $\gamma \in \Gamma$, the geodesic segment $[x, \gamma x]$ intersects at most $N \leq d(x, \gamma x)$ distinct horoballs $H \in \mathcal{H}$ and can be decomposed as

$$[x, \gamma x] = [x_0^+, x_1^-] \cup [x_1^-, x_1^+] \cup \cdots \cup [x_{N-1}^+, \gamma x_N^-]$$

where $x_0^+ = x$, $x_N^- = x$, and where $[x_i^-, x_i^+]$ is equal to $[x, \gamma x] \cap H_i$ for some horoball $H_i \in \mathcal{H}$ and each $[x_i^+, x_{i+1}^-]$ is included in $\widetilde{\mathcal{K}}$. Then, there exist elements $g_i \in \Gamma$ and $p_i \in P_1 \cup \cdots \cup P_l$, for $1 \leq i \leq N-1$, with $g_N := \gamma$, such that $x_i^- \in g_i\mathcal{K}$, $x_i^+ \in g_i p_i \mathcal{K}$; moreover, set $\gamma_i := p_{i-1}^{-1} g_{i-1}^{-1} g_i$ for $1 \leq i \leq N$ with the convention $p_0 = g_0 = 1$.

Notice that all the geodesics $[x, \gamma_i x]$ are included in a $D' = D'(D)$-neighbourhood of $\widetilde{\mathcal{K}}$: in fact, the length of the broken geodesic $[x_i^+, g_i p_i x] \cup [g_i p_i x, g_{i+1} x] \cup [g_{i+1} x, x_{i+1}^-]$ exceeds the length of $[x_i^+, x_{i+1}^-]$ at most of $2D$, so (the curvature being bounded above by -1) it stays $D'(D)$ close to $[x_i^+, x_{i+1}^-]$; by construction this last segment does not enter any horoball of $\mathcal{H}$, so $[g_i p_i x, g_{i+1} x]$ and $[x, \gamma_{i+1} x] = p_i^{-1} g_i^{-1} [g_i p_i x, g_{i+1} x]$ stay in the D'-neighbourhood of $\widetilde{\mathcal{K}}$.

Now we have $\gamma = \gamma_1 p_1 \gamma_2 \cdots \gamma_{N-1} p_{N-1} \gamma_N$, so

$$d_0(x_0, \gamma x_0) \le \sum_{i=1}^{N} d_0(x_0, \gamma_i x_0) + \sum_{i=1}^{N-1} d_0(x_0, p_i x_0). \tag{6}$$

On the other hand

$$\begin{aligned} d(x, \gamma x) &= \sum_{i=1}^{N} d(x_{i-1}^+, x_i^-) + \sum_{i=1}^{N-1} d(x_i^-, x_i^+) \\ &\ge \sum_{i=1}^{N-1} d(g_{i-1} p_{i-1} x, g_i x) + \sum_{i=1}^{N-1} d(g_i x, g_i p_i x) - 4(N-1)D \\ &= \sum_{i=1}^{N} d(x, \gamma_i x) + \sum_{i=1}^{N} d(x, p_i x) - 4(N-1)D \end{aligned}$$

which in turn yields, as $N \le d(x, \gamma x)$,

$$\sum_{i=1}^{N} d(x, \gamma_i x) + \sum_{i=1}^{N} d(x, p_i x) \le (1 + 4D) d(x, \gamma x). \tag{7}$$

To obtain inequality (5), it is thus sufficient to check that it holds for each γ_i and p_i which appears in the sums (6) and (7). This is proved in the two following lemmas:

Lemma 3.3 *For any $D' > 0$, there exists $C > 0$ such that*

$$d_0(x_0, \gamma x_0) \le C\, d(x, \gamma x) + C$$

for any $\gamma \in \Gamma$ such that $[x, \gamma x]$ lies in the D'-neighbourhood of $\widetilde{\mathcal{K}}$.

Lemma 3.4 *There exists $C' > 0$ such that, for any parabolic isometry $p \in P_1 \cup \cdots \cup P_l$*

$$d(x, px) \stackrel{C'}{\asymp} d_0(x_0, px_0). \tag{8}$$

Switching the roles of (X, d) and (X_0, d_0), we obtain the opposite inequality $d_0(x_0, \gamma x_0) \le \lambda d(x, \gamma x) + \lambda$, which concludes the proof of Proposition 3.2. □

Proof of Lemma 3.3 Let $\gamma \in \Gamma$ such that $[x, \gamma x]$ lies in the D'-neighbourhood of $\widetilde{\mathcal{K}}$, and let $x_0 = x$, $x_N = \gamma x$ and $x_1, \cdots, x_{N-1}$ be the points on the geodesic segment $[x, \gamma x]$ such that $d(x, x_i) = iD$ for $0 \le i \le N-1$, with $N - 1 = [d(x, \gamma x)]$. There exist isometries $h_0 = 1, h_1, \cdots, h_{N-1}, h_N = \gamma$ in Γ such that $d(x_i, h_i x) \le D + D'$; setting $k_i = h_{i-1}^{-1} h_i$, we then have $\gamma = k_1 k_2 \cdots k_N$. Now, for any $1 \le i \le N$, we have $d(x, k_i x) \le 1 + D + D'$; so every k_i belongs to the finite set $B := \{k \in \Gamma \mid d(x, kx) \le 1 + D + D'\}$. Setting $C := \max\{d_0(x_0, kx_0) \mid k \in B\}$, we obtain

$$d_0(x_0, \gamma x_0) \le \sum_{i=1}^{N} d_0(x_0, k_i x_0) \le NC \le C d(x, \gamma x) + C.$$

□

Proof of Lemma 3.4 Let us first notice that if $p \in \Gamma$ acts on X as a parabolic (resp. a hyperbolic) isometry, then it acts in the same way on X_0: actually, the infimum of the length of curves in $\bar{X}$ in the free homotopy class of a parabolic element p is 0 and this condition is preserved since $\bar{X}$ and $\bar{X}_0$ have the same length spectrum.

Then, let $\xi_1, ..., \xi_l \in X(\infty)$ be the fixed points of the maximal parabolic subgroups $P_1, .., P_l$ of Γ such that the geodesic rays $[x, \xi_i[$ are included in the Dirichlet domain $\mathcal{D}$, as described in the Sect. 2.1, and call ξ_i' the corresponding parabolic fixed points of $X_0(\infty)$; in order to simplify the notations, we set $P = P_i$, $\xi = \xi_i$ and $\xi' = \xi_i'$. Fix a finite generating set S for P and let $|\cdot|_S$ be the corresponding word metric. As P acts cocompactly by isometries on $(\partial H_\xi(x), d_\xi)$ and on $(\partial H_{\xi'}(x_0), d_{\xi'})$ we know that these metric spaces are both quasi-isometric to $(P, |\cdot|_S)$. In particular, there exists a constant $c > 0$ such that, for any $p \in P$

$$\frac{1}{c} d_\xi(x, px) - c \le d_{\xi'}(x_0, px_0) \le c d_\xi(x, px) + c. \tag{9}$$

Now, by the bounds on curvature $-b^2 \le K_X \le -1$ we get (cp. [21])

$$2 \sinh\left(\frac{d(x, px)}{2}\right) \le d_\xi(x, px) \le \frac{2}{b} \sinh\left(\frac{b}{2} d(x, px)\right)$$

hence $d(x, px)/d(x_0, px_0) \overset{C'}{\asymp} 1$ for a constant $C' > 0$ only depending on b and c. □

Proposition 3.5 *Let α and β be two hyperbolic isometries in Γ with, respectively, repelling and attractive fixed points $\alpha^-, \alpha^+, \beta^-, \beta^+$). Then*

$$\lim_{n \to +\infty} e^{l(\alpha^n) + l(\beta^n) - l(\beta^n \alpha^n)} = [\alpha^-, \beta^-, \alpha^+, \beta^+]$$

where $l(\gamma)$ denotes the length of the closed geodesic corresponding to γ for any $\gamma \in \Gamma$.

The set of couples (α^-, α^+) of all hyperbolic fixed points of Γ being dense in $X(\infty)^2$, this shows that $[f_\infty(\xi_1), f_\infty(\xi_2), f_\infty(\xi_3), f_\infty(\xi_4)] = [\xi_1, \xi_2, \xi_3, \xi_4]$ $\forall \xi_1, \xi_2, \xi_3, \xi_4 \in X(\infty)$.

Proof of Proposition 3.5 Fix $x \in X$. For $n \ge 0$, set $\gamma_n = \beta^n \alpha^n$, and let γ_n^-, γ_n^+ be its repelling and attractive fixed points. Consider two sequences of points $a_k \in]\alpha^-, \alpha^+[$ and $b_k \in]\beta^-, \beta^+[$ such that $\lim_{k \to +\infty} a_k = \alpha^-$ and $\lim_{k \to +\infty} b_k = \beta^+$; moreover, we can choose a sequence $n_k \to \infty$ such that $\lim_{k \to +\infty} \alpha^{n_k} a_k = \alpha^+$ and $\lim_{k \to +\infty} \beta^{-n_k} b_k = \beta^-$.

Now, for each k, let B_k be a compact ball centered at x containing a_k and b_k. Notice that γ_n^- and γ_n^+ tend respectively to α^- and β^+, so the sequence of geodesics $]\gamma_n^-, \gamma_n^+[$ tend to $]\alpha^-, \beta^+[$: namely, for k fixed, the distance between $]\gamma_n^-, \gamma_n^+[$ and $]\alpha^-, \beta^+[$, restricted to the compact set B_k, tends to 0 when $n \to \infty$. We can then choose n_k large enough so that

$$d\Big( ]\gamma_{n_k}^-, \gamma_{n_k}^+[\cap B_k \ , \ ]\alpha^-, \beta^+[\cap B_k \Big) < 1/k$$

Call a_k', b_k' the projections of a_k, b_k on $]\gamma_{n_k}^-, \gamma_{n_k}^+[$; so, the sequences $(a_k')_k$ and $(b_k')_k$ also converge respectively to α^- and β^+, and the sequences $(\alpha^{n_k} a_k')_k$, $(\beta^{-n_k} b_k')_k$ respectively to α^+ and β^-.

We then have:

$$[\alpha^-, \beta^-, \alpha^+, \beta^+] = \lim_{k \to \infty} \frac{e^{d(a_k', \alpha^{n_k} a_k') + d(\beta^{-n_k} b_k', b_k')}}{e^{d(a_k', b_k') + d(\beta^{-n_k} b_k', \alpha^{n_k} a_k')}} = \lim_{k \to \infty} \frac{e^{d(a_k, \alpha^{n_k} a_k) + d(\beta^{-n_k} b_k, b_k)}}{e^{d(a_k', b_k') + d(\beta^{-n_k} b_k', \alpha^{n_k} a_k')}} \tag{10}$$

by definition of the cross-ratio. Notice that the numerator gives exacty $e^{l(\alpha^{n_k})+l(\beta^{n_k})}$, as the points a_k and b_k lie on the axes of α, β respectively. On the other hand, for $k \gg 0$

$$d(a'_k, b'_k) + d(\beta^{-n_k} b'_k\ ,\ \alpha^{n_k} a'_k) = l(\gamma_{n_k}). \tag{11}$$

Indeed, let $V_\beta(b_k)$ be the hyperplane orthogonal to the axis of β, passing through b_k. When k is large enough, the point $\alpha^{n_k} a'_k$ is close to α^+, in particular it belongs to the half space bounded by $\beta^{-n_k}(V_\beta(b_k))$ which contains β^+; consequently, the point $\gamma_{n_k} a'_k = \beta^{n_k}\alpha^{n_k} a'_k$ belongs to the half-space $V_\beta(b_k)$ which contains β^+, so b'_k lies on the geodesic $(\gamma^-_{n_k}, \gamma^+_{n_k})$ between a'_k and $\gamma_{n_k} a'_k$. As $d(b'_k, \gamma_{n_k} a'_k) = d(\beta^{-n_k} b'_k\ ,\ \alpha^{n_k} a'_k)$, the equality (11) readily follows. Letting $k \to +\infty$ in (10) then achieves the proof. □

4 Entropy rigidity

This section is devoted to the proof of Theorem 1.2. The proof is through the method of barycenter, initiated by Besson-Courtois-Gallot [1], [2], and follows the lines of [10] (Theorem 1.6, holding for compact quotients). The main difficulty in the finite volume, non compact case is to show that the map produced by the barycenter method is proper: we will recall in Sect. 4.2 the main steps of the construction, referring the reader to [10] for the estimates which are now well established, while we will focus on the new estimates necessary to prove properness. For this, we will need accurates estimates of the Patterson-Sullivan measure of some subsets of $X(\infty)$, which we will describe in the first subsection.

4.1 On the Patterson measure of non uniform lattices

The Patterson-Sullivan measures of Γ are a family of finite measures $\mu = (\mu_x)$, indexed by points of X and with support included in the limit set $L\Gamma \subset X(\infty)$, satisfying the following properties (cp. for instance [36], [32] for details about their construction):

1. they are absolutely continuous w.r. to each other: for any $x, x' \in X$

$$\frac{d\mu_{x'}}{d\mu_x}(\xi) = e^{-\delta_\Gamma b_\xi(x',x)} \tag{12}$$

2. they are Γ-equivariant: for every $\gamma \in \Gamma$ and every Borel set $A \subset X(\infty)$

$$\mu_x(\gamma^{-1}A) = \mu_{\gamma x}(A) \tag{13}$$

When Γ is a lattice, we will use the decomposition of X explained in Sect. 2.1 to describe the local behavior of Patterson-Sullivan measures of Γ on the limit set $\Lambda_\Gamma = X(\infty)$.

For $x \in X$ and $\zeta \in X(\infty)$, we consider the point $x\zeta(R)$ at distance R from x on the geodesic ray $[x, \zeta[$, and define the "spherical cap" $V_\zeta(x, R) \subset X(\infty)$ as the set of points of $X(\infty)$ whose projection on the geodesic ray $[x, \zeta[$ falls between $x\zeta(R)$ and ζ. The proposition below gives an uniform estimate, which will be crucial in the sequel, for the measure $\mu_x(V_\zeta(x, R))$ of "small" spherical caps, i.e. when $R \gg 0$.
So, let $\mathcal{D} = \mathcal{K} \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_l$ be a decomposition of the Dirichlet domain of Γ centered at some fixed point x, corresponding to the maximal, bounded parabolic subgroups P_1, ..., P_l of Γ with fixed points ξ_1, ..., ξ_l as described in 2.1. If $x\zeta(R)$ projects to the thick part $\bar{\mathcal{K}}$ of $\bar{X}$, then formulas (12) and (13) easily give the uniform lower estimate:

$$\mu_x(V_\zeta(x, R)) \overset{c}{\succeq} e^{-\delta_\Gamma R} \tag{14}$$

(where c is a positive constant, depending on the minimal mass of a spherical cap at distance less than $D = diam(\mathcal{K})$ from x). On the other hand, when $x\zeta(R)$ projects to the cuspidal part, we have:

Proposition 4.1 *There exists a constant $c > 0$ satisfying the following property. Let $\zeta \in X(\infty)$ and assume that the point $x\zeta(R)$ belongs to $\gamma\mathcal{C}_i$, $R > 0$. Then:*

$$\mu_x(V_\zeta(x, R)) \overset{c}{\succeq} e^{-\delta_\Gamma(R+r)} v_{P_i}(x, 2r) \tag{15}$$

where $r = b_{\xi_i}(x, \gamma^{-1}x\zeta(R))$.

This estimate stems from a series of technical lemmas, and might be deduced from work developed in [31] and [34] (notice however that, in [31], μ_x has no atomic part, and in [34] the parabolic subgroups are assumed to satisfy an additional, strong regularity assumption). Since the estimate is of independent interest, we will report for completeness the proof of Proposition 4.1, in full generality, in the Appendix.

4.2 Entropy rigidity : proof of Theorem 1.2

Let $\bar{X} = \Gamma\backslash X$, fix a point $x_0 \in X$ and call for short $b_\xi(x) = b_\xi(x, x_0)$.

The function b_ξ is strictly convex if $K_X \leq -1 < 0$, since for every point y we have:

$$Hess_y\, b_\xi \geq g_y - (db_\xi)_y \otimes (db_\xi)_y \tag{16}$$

where g denotes the metric tensor of X; moreover, it is known that if the equality holds in (16) at every point y and for every direction ξ, then the sectional curvature is constant, and X is isometric to the hyperbolic space $\mathbb{H}^n$. The idea of the proof is to show that the condition $\delta_\Gamma = n - 1$ forces the equality in (16).

Recall that, for every measure μ on $X(\infty)$ whose support is not reduced to one point, we can consider its *barycenter*, denoted $bar[\mu]$, that is the unique point of minimum of the function $y \mapsto \mathcal{B}_\mu(y) = \int_{X(\infty)} e^{b_\xi(y)} d\mu(\xi)$ (notice that this is C^2 and strictly convex function, as $b_\xi(y)$ is). If $supp(\mu)$ is not a single point, it is easy to see that $\lim_{y\to\xi} \mathcal{B}_\mu(y) = +\infty$ for all $\xi \in X(\infty)$ cp. [10].

Consider now the map $F : X \to X$ defined by

$$F(x) = bar\left[e^{-b_\xi(x)}\mu_x\right] = argmin\left[y \mapsto \int_{X(\infty)} e^{b_\xi(y,x)} d\mu_x(\xi)\right]$$

where $(\mu_x)_x$ is the family of Patterson–Sullivan measures associated with the lattice Γ. In [10] it is proved that the map F satisfies the following properties:

a. F is equivariant with respect to the action of Γ, i.e. $F(\gamma x) = \gamma F(x)$;
b. F is C^2, with Jacobian:

$$|Jac_x F| \leq \left(\frac{\delta_\Gamma + 1}{n}\right)^n det^{-1}(k_x) \tag{17}$$

where $k_x(u, v)$ is the bilinear form on $T_x X$ defined as

$$k_x(u,v) = \frac{\int_{X(\infty)} e^{b_\xi(F(x),x))}\left[(db_\xi)^2_{F(x)} + Hess_{F(x)} b_\xi\right](u,v)\, d\mu_x(\xi)}{\int_{X(\infty)} e^{b_\xi(F(x),x))}\, d\mu_x(\xi)} \tag{18}$$

Notice that the eigenvalues of k_x are all greater or equal than 1, by (16).
Property (a) stems from the equivariance (i) of the family of Patterson-Sullivan measures with respect to the action of Γ, and from the cocycle formula for the Busemann function: $b_\xi(x_0, x) + b_\xi(x, y) = b_\xi(x_0, y)$. Property (b) comes from the fact that the Busemann function is C^2 on Hadamard manifolds, and is proved by direct computation, which does not use cocompactness.

By equivariance, the map F defines a quotient map $\bar{F} : \bar{X} \to \bar{X}$, which is homotopic to the identity through the homotopy

$$\bar{F}_t(x) = bar\left[e^{-b_\xi(x)}\,(t\mu_x + (1-t)\lambda_x)\right] mod\ \Gamma, \quad t \in [0,1]$$

where λ_x is the visual measure from x (with total mass equal to the volume of S^{n-1}).
Actually, the map $F_t = bar\left[e^{-b_\xi(x)}\,(t\mu_x + (1-t)\lambda_x)\right]$ defines a map between the quotient manifolds, as it is still Γ-equivariant; moreover, we have $bar\left[e^{-b_\xi(x)}\lambda_x\right] = x$ since, for all $v \in T_x X$:

$$\left(d\mathcal{B}_{e^{-b_\xi(x)}\lambda_x}\right)_x(v) = \int_{X(\infty)} (db_\xi)_x(v) e^{b_\xi(x)} e^{-b_\xi(x)} d\lambda_x(\xi) = \int_{U_x X} g_x(u,v)du = 0.$$

We will now prove that:

Proposition 4.2 *The homotopy map $\bar{F}_t$ is proper.*

Assuming for a moment Proposition 4.2, the proof of Theorem 1.2 follows by the degree formula: since $\bar{F}$ is properly homotopic to the identity, it has degree one, so

$$\begin{aligned}
\mathrm{vol}(\bar{X}) = \left|\int_{\bar{X}} \bar{F}^* dv_g\right| &\le \int_{\bar{X}} |Jac_x \bar{F}| dv_g \\
&\le \left(\frac{\delta_\Gamma + 1}{n}\right)^n \int_{\bar{X}} det^{-1}(k_x) dv_g \\
&\le \left(\frac{\delta_\Gamma + 1}{\delta_\Gamma(\mathbb{H}^n) + 1}\right)^n \mathrm{vol}(\bar{X})
\end{aligned}$$

as $det(k_x) \ge 1$ everywhere. So, if $\delta_\Gamma = \delta_\Gamma(\mathbb{H}^n) = n-1$, we deduce that $det(k_x) = 1$ everywhere and $k = g$, hence the equality in the equation (16) holds for every $y = F(x)$ and ξ. Since F is surjective, this shows that X has constant curvature -1. □

Proof of Proposition 4.2. Denote by $\bar{z}$ the projection of a point $z \in X$ to $\bar{X}$, and set $\delta = \delta_\Gamma$; recall that $\delta = n-1$, but we will use this property only at the end of the proof.

Let $\mu_x^t = e^{-b_\xi(x)}\,(t\mu_x + (1-t)\lambda_x)$: we need to show that if $t_k \to t_0$ and if $\bar{x}_k \to \infty$ in $\bar{X}$, then $\bar{y}_k = \bar{F}_{t_k}(\bar{x}_k) = \overline{bar[\mu_{x_k}^{t_k}]}$ goes to infinity too.

Now, assume by contradiction that the points $\bar{y}_k$ stay in a compact subset of $\bar{X}$: so (up to a subsequence) $\bar{x}_k, \bar{y}_k$ lift to points x_k, y_k such that $y_k \to y_0 \in X$ and $d(y_0, x_k) = d(\bar{y}_0, \bar{x}_k) = R_k \to \infty$.

By the cocycle relation $b_\xi(y,x) = b_\xi(y,y_0) + b_\xi(y_0,x)$ and by the density formula for the Patterson-Sullivan measures $\frac{d\mu_x}{d\mu_{y_0}}(\xi) = e^{-\delta b_\xi(x,y_0)}$, we have

$$(d\mathcal{B}_{\mu_x^t})_y(v) = t\int_{X(\infty)} (db_\xi)_y(v)\, e^{b_\xi(y,y_0)} e^{(\delta+1)b_\xi(y_0,x)} d\mu_{y_0}(\xi)$$
$$+(1-t)\int_{X(\infty)} (db_\xi)_y(v) e^{b_\xi(y,x)}\, d\lambda_x(\xi) \tag{19}$$

We will now estimate the two terms in (19) and show that $(d\mathcal{B}_{\mu_{x_k}^{t_k}})_{y_k}$ does not vanish for $R_k \gg 0$, a contradiction. So, let ζ_k be the endpoints of the geodesic rays y_0x_k and let $v_k = (\nabla b_{\zeta_k})_{y_k}$. Also, consider the spherical caps $V_{\zeta_k}(y_0, R_k/2)$ and $V_{\zeta_k}(y_0, R_k)$.
Let us first consider the contributions of the two integrals of the right hand side in (19) over $X \setminus V_{\zeta_k}(y_0, R_k/2)$. If $\xi \in X(\infty) \setminus V_{\zeta_k}(y_0, R_k/2)$, the projection of ξ over $y_0\zeta_k$ falls closer to y_0 than to x_k, hence $b_\xi(y_0, x_k) \le 0$; moreover, $|b_\xi(y_k, y_0)| \le d(y_k, y_0) \to 0$, so the first integral on $X \setminus V_{\zeta_k}(y_0, R_k/2)$ for $x = x_k$, $y = y_k$ and $v = v_k$ gives:

$$\left| \int_{X\setminus V_{\zeta_k}(y_0,\frac{R_k}{2})} (db_\xi)_{y_k}(v_k) e^{b_\xi(y_k,y_0)} e^{(\delta+1)b_\xi(y_0,x_k)}\, d\mu_{y_0} \right| \le 2 \parallel \mu_{y_0} \parallel$$

for $k \gg 0$. Analogously, the second integral on $X \setminus V_{\zeta_k}(y_0, R_k/2)$ yields

$$\left| \int_{X\setminus V_{\zeta_k}(y_0,\frac{R_k}{2})} (db_\xi)_{y_k}(v_k) e^{b_\xi(y_k,x_k)} d\lambda_{x_k} \right| \le 2\,\mathrm{vol}(\mathbb{S}^{n-1})$$

for $k \gg 0$, since $|b_\xi(y_k, x_k) - b_\xi(y_0, x_k)| \le d(y_k, y_0)$. So, these contributions are bounded.
We now compute the contributions of the integrals over $V_{\zeta_k}(y_0, R_k/2) \setminus V_{\zeta_k}(y_0, R_k)$. For all $\xi \in V_{\zeta_k}(y_0, R_k/2)$ we have that $(\nabla b_\xi)_{y_0}(\nabla b_{\zeta_k})_{y_0}$ is close to 1, for $R_k \gg 0$; moreover, as

$$\left|(\nabla b_\xi)_{y_k} v_k - (\nabla b_\xi)_{y_0}(\nabla b_{\zeta_k})_{y_0}\right| \stackrel{k\to\infty}{\longrightarrow} 0,$$

we deduce that $(db_\xi)_{y_k}(v_k) > \frac{1}{2}$ on $V_{\zeta_k}(y_0, R_k/2)$ for $k \gg 0$, hence these contributions are positive.

Finally, let us compute the contributions of these integrals on the caps $V_{\zeta_k}(y_0, R_k)$. For $\xi \in V_{\zeta_k}(y_0, R_k)$, consider the ray $y_0\xi$ from y_0 to ξ, and the projection $P(t)$ on the geodesic $y_0\zeta_k$ of the point $\xi(t) := y_0\xi(t)$. We have, by (1)

$$b_\xi(y_0, x_k) \ge \lim_{t\to\infty} [d(y_0, P(t)) + d(P(t), \xi(t))]$$
$$-[d(\xi(t), P(t)) + d(P(t), x_k)] - \epsilon 0 \ge R_k - \epsilon$$

with $\epsilon = \epsilon(\pi/2)$. Therefore we deduce that, for $k \gg 0$, we have

$$\int_{V_{\zeta_k}(y_0,R_k))} (db_\xi)_{y_k}(v_k) e^{b_\xi(y_k,y_0)} e^{(\delta+1)b_\xi(y_0,x_k)}\, d\mu_{y_0} \ge \frac{1}{4} e^{(\delta+1)(R_k-\epsilon)} \mu_{y_0}(V_{\zeta_k}(y_0, R_k)) \tag{20}$$

$$\int_{V_{\zeta_k}(y_0,R_k))} (db_\xi)_{y_k}(v_k) e^{b_\xi(y_k,x_k)}\, d\lambda_{x_k} \ge \frac{1}{4} e^{(R_k-\epsilon)}\, \mathrm{vol}(\mathbb{S}^{n-1}). \tag{21}$$

It is clear that the right-hand side of (21) goes to infinity when $R_k \gg 0$; we will now prove that the right-hand side of (20) also diverges for $R_k \to \infty$. This will conclude the proof, as it will show that $d\mathcal{B}_{\mu_{x_k}^{t_k}}(v_k)$ does not vanish for $k \gg 0$ (being a convex combination of two positively diverging terms).

So, let $\mathcal{D} = \mathcal{K} \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_l$ be a decomposition of the Dirichlet domain of Γ centered at y_0 as in (2), corresponding to maximal, bounded parabolic subgroups P_1, ..., P_l with fixed points ξ_1, ..., ξ_l. We know that $\bar{x}_k$ belongs to some cusp of $\bar{X}$, so $x_k \in \gamma \mathcal{C}_i$ for some γ; let then $r_k = b_{\xi_i}(y_0, \gamma^{-1} x_k) \leq R_k$.

If $\gamma \xi_i$ falls in $V_{\zeta_k}(y_0, R_k)$ and $\delta \gg 0$, as $K_X \geq -b^2$ we can use Propositions 4.1 and 2.3 to deduce that

$$\begin{aligned} e^{(\delta+1)R_k} \mu_{y_0}(V_{\zeta_k}(y_0, R_k)) &\succeq e^{(\delta+1)R_k} e^{-\delta(R_k + r_k)} \sum_{n \geq 0} v_{P_i}(2r) e^{-\delta n} \\ &\succeq e^{R_k - \delta r_k} v_{P_i}(2r_k) \\ &\succeq \frac{e^{R_k - \delta r_k}}{\mathcal{A}_{P_i}(x_0, y_0, r_k)}. \end{aligned}$$

Since $K_X \leq -1$, we know that $\mathcal{A}_{P_i}(x_0, y_0, r_k) \preceq e^{-(n-1)r_k}$, so we obtain

$$e^{(\delta+1)R_k} \mu_{y_0}(V_{\zeta_k}(y_0, R_k)) \succeq e^{R_k + (n-1-\delta)r_k}.$$

On the other hand, when $\gamma \xi_i \notin V_{\zeta_k}(y_0, R_k)$, we have, by Propositions 4.1 and 2.3:

$$e^{(\delta+1)R_k} \mu_{y_0}(V_{\zeta_k}(y_0, R_k)) \ \succeq \ e^{R_k - \delta r_k} v_P(y_0, 2r_k) \succeq \ e^{R_k + (n-1-\delta)r_k}.$$

Both lower bounds tend to $+\infty$ as $k \to +\infty$, since $R_k \to +\infty$ and $\delta \leq n-1$; thus, the integral in (20) diverges. This concludes the proof that the map $\bar{F}_t$ is proper. □

5 Appendix

We report here, for completeness, a proof of the estimate given in Proposition 4.1.
To prove Proposition 4.1, we will need a series of elementary lemmas, where some equalities hold up to some constant: so we will use the symbol $f \overset{C}{\approx} g$ to mean that two quantities f and g differ of at most C. To avoid cumbersome notations, we will give the same name to these constants in all the lemmas, meaning that they all hold for the choice of a suitable constant C large enough. All these constants will depend on the upper bound of the sectional curvature $K_X \leq -1$ and, possibly, on other parameters of $\bar{X} = \Gamma \backslash X$ which we will specify case by case.

Recall that a parabolic group P of isometries fixing $\xi \in X(\infty)$ is called *bounded* if it acts cocompactly on $X(\infty) - \{\xi\}$ (as well as on every horosphere ∂H centered at ξ). If $\mathcal{D}(P, x)$ is the Dirichlet domain of P centered at x, the sets $\mathcal{S}_x = \mathcal{D}(P, x) \cap \partial H_\xi(x)$ and the trace at infinity $\mathcal{S}_x(\infty) = \overline{\mathcal{D}(P, x)} \cap X(\infty)$ of $\mathcal{D}(P, x)$ are compact, fundamental domains for the action of P on $\partial H_\xi(x)$ and on $X(\infty)$, respectively.
The following Lemmas can be found, for instance, in [34] (Lemmes 2.6, 2.7 and 2.9):

Lemma 5.1 *There exists a constant $C > 0$ with the following property.*
Let $x \in X$ and $\zeta \in X(\infty)$ be fixed. Then, for any $\xi \in V_\zeta(x, R)$ we have:

$$d(x\zeta(R), x\xi(R)) \leq C.$$

Lemma 5.2 *There exists a constant $C > 0$ with the following property.*
Let $x \in X$ and $\zeta \in X(\infty)$ be fixed. Then:
(i) for any x' such that $d(x, x') < C$ we have

$$V_\zeta(x', R + C) \subset V_\zeta(x, R) \subset V_\zeta(x', R - C)$$

(ii) for any $\xi \in V_\zeta(x, R+2C)$ *we have*

$$V_\xi(x, R+C) \subset V_\zeta(x, R) \subset V_\xi(x, R-C)$$

provided that $R > C$.

Lemma 5.3 *Let* P *a bounded parabolic subgroup of* X *fixing* ξ*, and let* $S_x(\infty)$ *as above. There exists a constant* $C > 0$ *(depending on the diameter of* $\mathcal{S}_x$*) with the following properties: for any* $p \in P$
(i) if $d(x, px) > 2R$*, with* $R > C$*, then* $\forall \eta \in S_x(\infty)$ *we have* $p\eta \in V_\xi(x, R-C)$ *and*

$$b_{p\eta}(x_R, px_R) \stackrel{C}{\approx} d(x, px) - 2R$$

(ii) if $d(x, px) \leq 2R$*, then* $\forall \eta \in S_x(\infty)$ *we have* $p\eta \in X(\infty) \setminus V_\xi(x, R+C)$ *and*

$$b_{p\eta}(x_R, px_R) \leq C$$

where $x_R = x\xi(R)$ *is the point at distance* R *from* x *on the geodesic ray* $[x, \xi[$.

Proof of Proposition 4.1. Let $\mathcal{D} = \mathcal{K} \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_l$ be a decomposition of the Dirichlet domain of Γ centered at x, corresponding to the maximal, bounded parabolic subgroups P_1, ..., P_l of Γ with parabolic fixed points ξ_1, ..., ξ_l, and with $\mathcal{C}_i = \mathcal{D} \cap H_{\xi_i}$, as described in 2.1. Moreover, let $\mathcal{S}_i(\infty) = \overline{\mathcal{D}} \cap X(\infty)$ be the fundamental domains for the action of P_i on $X(\infty) \setminus \{\xi_i\}$, and let $z_t := x\zeta(t)$ and $x_{i,t} := x\xi_i(t)$.

We assume that z_R belongs to γH_{ξ_i}; so, call for short $\xi = \xi_i$, $P = P_i$, $x_R = x_{i,R}$, $\mathcal{S}(\infty) = \mathcal{S}_i(\infty)$ and set $r = b_\xi(x, \gamma^{-1} z_R)$ hereafter.

Now, first notice that $|b_\eta(x, z_R) - R|$ is bounded, uniformly in $\eta \in V_\zeta(x, R)$, since for $t \gg 0$ we have

$$b_\eta(x, z_R) \stackrel{\epsilon}{\approx} (d(x, z_R) + d(z_R, x\eta(t))) - d(x\eta(t), z_R) = R$$

for $\epsilon = \epsilon(\frac{\pi}{2})$ as in (1). Thus, the density formula (12) yields

$$\mu_x\big(V_\zeta(x, R)\big) \stackrel{c}{\asymp} e^{-\delta_\Gamma R} \mu_{z_R}\big(V_\zeta(x, R)\big)$$

for some constant $c > 0$ only depending on the upper bound of the curvature.
It is thus sufficient to show that

$$\mu_{z_R}\big(V_\zeta(x, R)\big) \succeq e^{-\delta_\Gamma r} v_P(x, 2r) \tag{22}$$

For this, we will analyse two different cases.

Case 1: $\zeta \in \Gamma\xi$.

a) Assume first $\gamma = 1$, so $\zeta = \xi$ and $z_R = x_R \in H_\xi$.
We have, by Lemma 5.3:

$$\begin{aligned}\mu_{x_R}\left(V_\xi(x, R)\right) &\geq \mu_{x_R}(\{\xi\}) + \sum_{\substack{p \in P \\ p\mathcal{S}(\infty) \subset V_\xi(x, R)}} \mu_{x_R}(p\mathcal{S}(\infty)) \\ &\geq \sum_{\substack{p \in P \\ d(x, px) \geq 2R + C}} \mu_{x_R}(p\mathcal{S}(\infty)).\end{aligned} \tag{23}$$

From the equivariance and the density formula (12), (13) for the family μ_x we get

$$\mu_{x_R}(p\mathcal{S}(\infty)) = \int_{\mathcal{S}(\infty)} e^{-\delta_\Gamma b_{p\eta}(x_R, px_R)} \mu_{x_R}(d\eta)$$
$$\asymp \mu_{x_R}(\mathcal{S}(\infty)) e^{\delta_\Gamma (2R - d(x,px))}$$

because $b_{p\eta}(x_R, px_R) \approx d(x, px) - 2R$, by Lemma 5.3.
Now, $\mu_{x_R}(\mathcal{S}(\infty)) \asymp e^{-\delta_\Gamma R} \mu_x(\mathcal{S}(\infty)) \asymp e^{-\delta_\Gamma R}$, since $X(\infty) = P\mathcal{S}(\infty) \cup \{\xi\}$ and the mass of μ_{x_R} is not reduced to one atom, so $\mu_x(\mathcal{S}(\infty)) > 0$.
Therefore, from (23) we deduce that, for Δ large enough, we have

$$\mu_{x_R}\big(V_\xi(x, R)\big) \;\succeq\; e^{\delta_\Gamma R} \sum_{\substack{p \in P_i \\ 2R+C+\Delta \geq d(x,px) \geq 2R+C}} e^{-\delta_\Gamma d(x,px)} \;\succeq\; e^{-\delta_\Gamma R} v_P(x, 2R) \tag{24}$$

as $v_P^\Delta(x, 2R) \asymp v_P(x, 2R)$ by Proposition 2.3, if $\Delta \geq \Delta_0$. The estimate (22) follows in this case, since $\zeta = \xi$ and $z_R = x_R$, so $r = b_\xi(x, z_R) = b_\xi(x, x_R) = R$.

b) Assume now that $\zeta = \gamma\xi$ for some $\gamma \neq 1$.
We then set $\xi' = \gamma\xi = \zeta$, $x' = \gamma x$, $H_{\xi'} = \gamma H_\xi$, $x'_t = \gamma x_t$ and $R' := b_{\xi'}(x', z_R)$.
Notice that, without loss of generality, we can assume that x' lies at distance less than $\mathrm{diam}(\mathcal{K})$ from the geodesic ray $x\xi'$ (actually, as P acts cocompactly on ∂H_ξ, we can replace γ by γp for some suitable $p \in P$), so $d(z_{R-R'}, x')$ and $d(z_R, x'_{R'})$ are both bounded by $2\mathrm{diam}(\mathcal{K})$.
By Lemma 5.2(i), there exists $C > 0$ such that

$$V_{\xi'}(x, R) = V_{\xi'}(z_{R-R'}, R') \supset V_{\xi'}(x', R' + C)$$

and then (22) follows from a), by applying the inequality (24) to ξ', x' and R'. Actually, as $d(z_R, x'_{R'})$ is bounded, we have $d\mu_{x'_{R'}}/d\mu_{z_R} \asymp 1$ and we get from (24)

$$\mu_{z_R}(V_{\xi'}(x, R)) \succeq \mu_{x'_{R'}}(V_{\xi'}(x', R' + C)) \succeq e^{-\delta_\Gamma R'} v_P(x', 2R')$$

which gives (22), since in this case $\xi' = \zeta$ and $r = b_\xi(x, \gamma^{-1} z_R) = b_{\xi'}(x', z_R) = R'$.

Case 2: $\zeta \notin \Gamma\xi$.
a) Assume first that $\gamma = 1$, so $z_R \in H_\xi$.
Let C be the constant in Lemma 5.2. If $\xi \in V_\zeta(x, R - 2C)$ we call $S = R - 4C$, so that $\xi \in V_\zeta(x, S + 2C)$ and we have $V_\zeta(x, S) \supset V_\xi(x, S + C)$ by Lemma 5.2(ii). Notice that we have $d(z_R, x_S) \leq 5C$ by Lemma 5.1. Therefore, applying again (24) to ξ, x and S, we get

$$\mu_{z_R}\big(V_\zeta(x, S)\big) \succeq \mu_{x_S}\big(V_\xi(x, S + C)\big) \succeq e^{-\delta_\Gamma S} v_P(x, 2S) \tag{25}$$

and the estimate (22) follows, since here $r = b_\xi(x, z_R) \approx b_\xi(x, x_S) = S$.
On the other hand, if $\xi \notin V_\zeta(x, R - 2C)$, let $\bar{\zeta}$ be the point at infinity of the geodesic supporting $]\zeta, x]$, different from ζ, and let $\bar{x}$ be the point of $]\zeta, x] \cap \partial H_\xi$ closest to ζ.
Moreover, let $\bar{R} := d(\bar{x}, z_R)$. Notice that $z_R = \bar{x}\bar{\zeta}(\bar{R})$ and that, setting $\bar{x}_{\bar{R}} := \bar{x}\xi(\bar{R})$, we have $d(z_R, \bar{x}_{\bar{R}}) < C$, always by Lemma 5.1, so $d\mu_{\bar{x}_{\bar{R}}}/d\mu_{z_R} \asymp 1$.

Now, we have $V_\zeta(x, R) = X(\infty) \setminus V_{\bar\zeta}(\bar x, \bar R)$ and $X(\infty) \setminus V_\zeta(x, R-2C) = V_{\bar\zeta}(\bar x, \bar R+2C)$; as $\xi \notin V_\zeta(x, R-2C)$ we deduce that $V_\zeta(x, R) \supset X(\infty) \setminus V_\xi(\bar x, \bar R - C)$ by Lemma 5.2(ii). Hence,

$$\mu_{z_R}(V_\zeta(x, R)) \succeq \mu_{\bar x_{\bar R}}\Big(X(\infty) \setminus V_\xi(\bar x, \bar R - C)\Big). \tag{26}$$

Similarly to case 1, we can estimate this by applying Lemma 5.3 to $\bar x$ and ξ :

$$\begin{aligned}\mu_{\bar x_{\bar R}}\Big(X(\infty) \setminus V_\xi(\bar x, \bar R - C)\Big) &\geq \sum_{\substack{p\in P\\ p\mathcal{S}(\infty)\cap V_\xi(\bar x,\bar R-C)=\emptyset}} \mu_{\bar x_{\bar R}}(p\mathcal{S}(\infty))\\ &\geq \sum_{\substack{p\in P\\ d(\bar x,p\bar x)\leq 2(\bar R-2C)}} \mu_{\bar x_{\bar R}}(p\mathcal{S}(\infty))\\ &\asymp \sum_{\substack{p\in P\\ d(\bar x,p\bar x)\leq 2(\bar R-2C)}} \mu_{\bar x_{\bar R}}(\mathcal{S}(\infty)) \end{aligned} \tag{27}$$

as $\mu_{\bar x_{\bar R}}(p\mathcal{S}(\infty)) \asymp \mu_{\bar x_{\bar R}}(\mathcal{S}(\infty))$ because, by Lemma 5.3 (ii),

$$\begin{aligned} b_{p\eta}(\bar x_{\bar R}, p\bar x_{\bar R}) &= b_{p\eta}(\bar x_{\bar R}, \bar x_{\bar R-2C}) + b_{p\eta}(\bar x_{\bar R-2C}, p\bar x_{\bar R-2C}) + b_{p\eta}(p\bar x_{\bar R-2C}, p\bar x_{\bar R})\\ &\leq 4C + b_{p\eta}(\bar x_{\bar R-2C}, p\bar x_{\bar R-2C})\\ &\leq 5C\end{aligned}$$

if $d(\bar x, p\bar x) \leq 2(\bar R - 2C)$. Moreover, $d(z_R, \bar x_{\bar R})$ is bounded, so we deduce that $\mu_{\bar x_{\bar R}}(\mathcal{S}(\infty)) \asymp \mu_{z_R}(\mathcal{S}(\infty)) \asymp e^{-\delta_\Gamma R}$ and that

$$\bar R = b_\xi(\bar x, \bar x_{\bar R}) \approx b_\xi(\bar x, z_R) = d(\partial H_\xi, \partial H_\xi(z_R)) \approx b_\xi(x, z_R) = r\ ;$$

so, combining (26) and (27) we obtain

$$\mu_{z_R}(V_\zeta(x, R)) \succeq e^{-\delta_\Gamma R} v_P(\bar x, 2\bar R) \succeq e^{-\delta_\Gamma r} v_P(x, 2r) \tag{28}$$

(since $\bar x$ is at bounded distance from the orbit of x).

b) Assume now that $\gamma \neq 1$.
We set $\xi' = \gamma\xi$, $x' = \gamma x$, $H_{\xi'} = \gamma H_\xi$, $x'_R = \gamma x_R$, with $d(x', [x, \xi[) \leq diam(\mathcal{K})$, and we proceed as above, according to the cases $\xi' \in V_\zeta(x, R-2C)$ or $\xi' \notin V_\zeta(x, R-2C)$.

In the first case, we call $S := R - 4C$, $S' = b_{\xi'}(x', z_S)$, so $V_\zeta(x, S) \supset V_{\xi'}(x, S + C)$ and we have $d(z_R, x'_{S'}) \leq 6C + 2diam(\mathcal{K})$; then, using Lemma 5.2, we deduce similarly to (25), that

$$\mu_{z_R}\big(V_\zeta(x, S)\big) \succeq \mu_{x'_{S'}}\big(V_{\xi'}(x', S' + 2C)\big) \succeq e^{-\delta_\Gamma S'} v_P(x', 2S')$$

which yields (22), as here $r = b_\xi(x, \gamma^{-1} z_R) \approx b_{\xi'}(x', z_S) = S'$.

In the second case, we call again $\bar\zeta$ the point at infinity opposite to ζ with respect to x, $\bar x$ the point of $]\zeta, x] \cap \partial H_{\xi'}$ closest to ζ, and we set $\bar R := d(\bar x, z_R)$, $\bar x_{\bar R} := \bar x\xi'(\bar R)$. So, $z_R = \bar x\bar\zeta(\bar R)$, $d(z_R, \bar x_{\bar R}) < 2C$ and $d\mu_{\bar x_{\bar R}}/d\mu_{z_R} \asymp 1$. Then, we deduce as before that $V_\zeta(x, R) \supset X(\infty) \setminus V_{\xi'}(\bar x, \bar R + C)$ and we obtain, analogously to (28), that

$$\mu_{z_R}(V_\zeta(x, R)) \succeq e^{-\delta_\Gamma R} v_P(\bar x, 2\bar R)$$

which concludes the proof as, in this case, $\bar R = b_{\xi'}(\bar x, \bar x_{\bar R}) \approx b_{\xi'}(x', z_R) = r$. □

References

1. Besson, G., Courtois, G., Gallot, S.: Entropies et rigidités des espaces localement symétriques de courbure strictement négative. Geom. Funct. Anal. **5**(5), 731–799 (1995)
2. Besson, G., Courtois, G., Gallot, S.: Lemme de Schwarz réel et applications géométriques. Acta Math. **183**(2), 145–169 (1999)
3. Boland, J., Connell, C., Souto, J.: Volume rigidity for finite volume manifolds. Am. J. Math. **127**(3), 535–550 (2005)
4. Bonk, M., Kleiner, B.: Rigidity for quasi-Möbius group actions. J. Differ. Geom. **61**(1), 81–106 (2002)
5. Bourdon, M.: Structure conforme au bord et flot géodésique d'un CAT(-1)-espace. Enseign. Math. **41**(1–2), 63–102 (1995)
6. Bourdon, M.: Sur le birapport au bord des $CAT(-1)$-espaces. Pub. Math. **83**, 95–104 (1996)
7. Bowditch, B.H.: Geometrical finiteness with variable negative curvature. Duke Math. J. **77**, 229–274 (1995)
8. Bowen, R.: Topological entropy for non-compact sets. Trans. Am. Math. Soc. **184**, 125–136 (1973)
9. Castillon, P., Sambusetti, A.: On asymptotically harmonic manifolds of negative curvature. Math. Zeit. **277**(3–4), 1049–1072 (2014)
10. Courtois G.: Critical exponents and rigidity in negative curvature, in Géométries à courbure négative ou nulle, groupes discrets et rigidités, 293–319, Sémin. Congr., 18, Soc. Math. France, Paris (2009)
11. Croke, C.B.: Rigidity for surfaces of non-positive curvature. Comment. Math. Helvetici **65**, 150–169 (1990)
12. Croke, C., Kleiner, B.: Conjugacy and rigidity for manifolds with a parallel vector field. J. Differ. Geom. **39**, 659–680 (1994)
13. Dal'bo, F., Otal, J.P., Peigné, M.: Séries de Poincaré des groupes géométriquement finis. Israel J. Math. **118**, 109–124 (2000)
14. Dal'bo, F., Peigné, M., Picaud, J.C.: Sambusetti A., On the growth of non-uniform lattices in pinched negatively curved manifolds, J. für die Reine und. Angew. Math. **627**, 31–52 (2009)
15. Dal'bo, F., Peigné, M., Picaud, J.C., Sambusetti, A.: Convergence and counting in infinite measure. Annales de l'Institut Fourier **67**(2), 483–520 (2017)
16. Dal'bo F., Peigné M., Picaud J.C., Sambusetti A.: Asymptotic geometry of negatively curved manifolds of finite volume, to appear in Annales Scientifiques de l' E.N.S. (2018)
17. Eberlein, P.: Geometry of nonpositively curved manifolds, Chicago Lectures in Mathematics
18. Hamenstädt, U.: Time preserving conjugacies of geodesic flows. Erg. Theory Dyn. Sys. **9**, 455–464 (1989)
19. Hamenstädt, U.: Entropy-rigidity of locally symmetric spaces of negative curvature. Ann. Math. **131**(1), 35–51 (1990)
20. Hamenstädt, U.: Cocycles, symplectic structures and intersection. Geom. Funct. Anal. **9**, 90–140 (1999)
21. Heintze, E., Im Hof, H.C.: Geometry of horospheres. J. Diff. Geom. **12**, 481–491 (1977)
22. Katok, A.: Entropy and closed geodesics. ETDS **2**, 339–367 (1982)
23. Kim, I.: Marked length rigidity of rank one symmetric spaces and their product. Topology **40**, 1295–1323 (2001)
24. Knieper, G.: Spherical means on compact Riemannian manifolds of negative curvature. Differ. Geom. Appl. **7**, 755–782 (1997)
25. Knieper, G.: On the asymptotic geometry of nonpositively curved manifolds. Geom. Funct. Anal. **7**, 755–782 (1997)
26. Manning, A.: Topological entropy for geodesic flows. Ann. Math. **110**, 567–573 (1979)
27. Otal, J.-P.: Le spectre marqué des longueurs des surfaces à courbure négative. Ann. Math. **131**(1), 151–162 (1990)
28. Otal, J.-P.: Sur la géométrie symlectique de l'èspace des géodésiques d'une variété à courbure négative. Rev. Mat. Iberoam. **8**(3), 441–456 (1990)
29. Otal, J.P., Peigné, M.: Principe variationnel et groupes kleiniens. Duke Math. J. **125**(1), 15–44 (2004)
30. Phan, T.: On finite volume, negatively curved manifolds, arXiv:1110.4087v2
31. Peigné M.: Autour de l'exposant de Poincaré d'un groupe kleinien, L'Enseignement Mathématique 43 (2013)
32. Roblin T.:Ergodicité et équidistribution en courbure négative, Mém. Soc. Math. Fr. (N.S.) no. 95 (2003)
33. Sambusetti, A.: Asymptotic properties of coverings in negative curvature. Geom. Topol. **12**(1), 617–637 (2008)
34. Schapira, B.: Lemme de l'ombre et non divergence des horosphères d'une variété géométriquement finie. Anna. Inst. Fourier **54**(4), 939–989 (2004)
35. Storm, P.: The minimal entropy conjecture for nonuniform rank one lattices. Geom. Funct. Anal. **16**(4), 959–980 (2006)

36. Sullivan, D.: The density at infinity of a discrete group of hyperbolic motions. IHES Publ. Math. **50**, 171–202 (1979)
37. Vigneras, M.F.: Varieés Riemanniennes isopectrales et non isométriques. Ann. Math. **112**, 21–32 (1980)